Non-simple purely infinite C*-algebras: the Hausdorff case

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ABSTRACT. Local and global definitions of pure infiniteness for a C^*-algebra A are compared, and equivalence between them is obtained if the primitive ideal space of A is Hausdorff and of finite dimension, if A has real rank zero, or if A is approximately divisible. Sufficient criteria are given for local pure infiniteness of tensor products. They yield that exact simple tensorially non-prime C^*-algebras are purely infinite if they have no semi-finite lower semi-continuous trace. One obtains that A is isomorphic to $A \otimes \mathcal{O}_\infty$ if A is (1-)purely infinite, separable, stable, nuclear and $\mathrm{Prim}(A)$ is a Hausdorff space (not necessarily of finite dimension).

1. Introduction

A major problem arising in the classification program for separable nuclear C^*-algebras is to detect the refined analogue for C^*-algebras of the type classification of von Neumann algebras introduced by Murray and von Neumann. Here we study the possible analogues of purely infinite (= type III) von Neumann algebras for C^*-algebras with Hausdorff primitive ideal space or for C^*-algebras with real rank zero, and extend some of the results of [44] and [45].

Given two non-zero positive elements a, b in a simple C^*-algebra A, one can find an integer n and a finite sequence $d_1, \ldots, d_n$ in A such that $\|b-(d_1^*ad_1+\ldots+d_n^*ad_n)\| < 1$. The simple C^*-algebra A is said to be *purely infinite* if one can always assume $n = 1$ in this relation, i.e., for all $a, b \in A_+ \setminus \{0\}$, there exists an operator $d \in A$ verifying the relation $\|b - d^*ad\| < 1$, and A is not equal to the complex numbers $\mathbb{C}$. We remind the reader in section 3 how he can easily see the equivalence of this definition to the original definition of purely infinite simple C^*-algebras by J. Cuntz on page 186 of [17].

Some notions of pure infiniteness for non-simple C^*-algebras have been recently introduced in [44], [45], [39] and [38] chap. 2, 3 (e.g. p.i. =pi(1), pi(n) with $n = 2, 3, \ldots$, strong pure infiniteness). It was shown in [45] that the definitions of pure infiniteness are equivalent in the cases of simple C^*-algebras, C^*-algebras of real rank zero and approximately divisible C^*-algebras. But in general it is not clear whether they coincide. Here we study the case of C^*-algebras with Hausdorff primitive ideal space.

The generalization of the notion p.i. to non-simple C^*-algebras is almost obvious:

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Definition 1.1. ([44]) A C^*-algebra A is said to be *purely infinite* (for short p.i.) if and only if

(i) for every pair of positive elements $a, b \in A_+$ such that b lies in the closed two-sided ideal $\overline{\mathrm{span}(AaA)}$ generated by a and for every $\varepsilon > 0$, there exists an element $d \in A$ such that $\|b - d^*ad\| < \varepsilon$, and
(ii) there is no non-zero character on A.

More natural is our notion of pi(n) for $n > 1$. It turns out to be equivalent to p. i. in the case of simple algebras, cf. section 3. To exclude sub-homogeneous algebras, we have to impose a generalization of assumption (ii) on $\ell_\infty(A)$, if A is non-simple. It reduces to the corresponding condition on A itself if A is unital, and it is always satisfied if A is stable.

Definition 1.2. ([38]) Given a strictly positive integer m, a C^*-algebra A is said to be *m-purely infinite* (abbreviated, pi(m)) if and only if

(i) for every pair of positive elements a, b in A, such that b lies in the closed two-sided ideal of A generated by a, and for every $\varepsilon > 0$, there exists $d_1, \ldots, d_m \in A$ such that $\|b - \sum_{1 \le i \le m} d_i^* a d_i\| < \varepsilon$, and
(ii) there is no non-zero quotient algebra of $\ell_\infty(A)$ of dimension $\le m^2$.

We say that A is *weakly purely infinite* (for short w.p.i.) if A is pi(m) for some $m \in \mathbb{N}$.

The property pi(m) passes to non-zero hereditary C^*-subalgebras and quotients of A, see Proposition 4.10. We do not know if we can replace $\ell_\infty(A)$ by A in (ii), but by Proposition 4.12 our Definition 1.2 implies the existence of $n \ge m$ such that A is n-purely infinite in the sense of [45, def. 4.3], i.e., for every $a \in A_+ \setminus \{0\}$, the element $a \otimes 1_n \in M_n(A)$ is properly infinite (cf. [44, def. 2.3] or Remark 2.9(ii)). In particular we work with the same notion of "weakly purely infinite" C^*-algebras as defined in [45, def. 4.3]. This implies that also the multiplier algebra $\mathcal{M}(A)$ of A is w.p.i., cf. [45, prop. 4.11]. Thus, $\mathcal{M}(A)$ has no quotient algebra of finite dimension, if A is pi(m). Conversely $\ell_\infty(\mathcal{M}(A))$ and, therefore, its ideal $\ell_\infty(A)$ can not have quotient algebras of dimension $\le m^2$, if $\mathcal{M}(A)$ has no quotient algebra of dimension $\le m^2$. Thus, with Proposition 4.12 and [45, prop. 4.11] in hand, we can replace (ii) equivalently by the requirement that $\mathcal{M}(A)$ *has no quotient algebra of dimension* $\le m^2$.

We characterize the C^*-algebras with Hausdorff primitive ideal space which have purely infinite simple quotients with help of the following local condition.

Definition 1.3. A C^*-algebra A is said to be *locally purely infinite* (abbreviated, *l.p.i.*) if and only if, for every primitive ideal J of A and every element $b \in A_+$ with $\|b+J\| > 0$, there is a non-zero stable C^*-subalgebra D of the hereditary C^*-subalgebra generated by b, such that D is not included in J.

We say that A is *traceless* if every lower semi-continuous non-negative 2-quasi-trace (cf. [27]) on A_+ is *trivial*, i.e., takes only the values 0 and $+\infty$. It turns out that locally purely infinite algebras are traceless, cf. Proposition 4.1. In particular, they must be anti-liminal.

Since approximately divisible C^*-algebras in the sense of [44, def. 5.5] are purely infinite by [44, thm. 5.9] if they are traceless, we can conclude from [45, prop. 5.14] that all sorts of pure infiniteness coincide on the class of approximately divisible C^*-algebras.

In section 3 we prove that locally purely infinite *simple* C^*-algebras are purely infinite and give some sufficient conditions under which spatial tensor products $A \otimes B$ are locally purely infinite. Then we use this to give a simple proof that *traceless exact simple tensorially non-prime C^*-algebras are purely infinite*. Another corollary is that $A \otimes C^*_r(F_2)$ is locally purely infinite if and only if every hereditary C^*-subalgebra D of the C^*-algebra A has only zero bounded (linear) traces.

Recently M. Rørdam [59] constructed an example of a simple nuclear C^*-algebra which contains both a properly infinite projection and a non-zero finite projection. This nuclear C^*-algebra is traceless and can not be purely infinite. Thus, "traceless" C^*-algebras are in general not locally purely infinite, even in the nuclear case.

A C^*-algebra A of real rank zero is locally purely infinite if and only if A is strongly purely infinite in the sense of the following Definition 1.4 (see Theorem 4.17).

Definition 1.4. ([45]) A C^*-algebra A is said to be *strongly purely infinite* (for short s.p.i.) if and only if for every $a, b \in A_+$, $\varepsilon > 0$, there exist elements $s, t \in A$ such that

$$\|a^2 - s^*a^2s\| < \varepsilon\,,\ \|b^2 - t^*b^2t\| < \varepsilon \ \text{ and } \|s^*abt\| < \varepsilon\,, \tag{1.2}$$

This definition is equivalent to [45, def. 5.1] by [45, rem. 5.10]. One can always assume the operators s, t to be contractions, cf. [45, cor. 7.22]. The property of strongly pure infiniteness for A passes to quotients A/J, hereditary C^*-subalgebras D of A, stabilizations and inductive limits, see [45, prop. 5.11], and from [45, cor. 7.22] it follows that s.p.i. passes also to $\ell_\infty(A)$ and to ultrapowers A_ω of A.

We obtain in section 5 that C^*-algebras A with Hausdorff primitive ideal space of finite topological dimension are locally purely infinite if and only if all its simple quotients are purely infinite, and that this is the case if and only if A is strongly purely infinite in the sense of Definition 1.4. The idea of the proof goes as follows: if the primitive ideal space of a C^*-algebra A is Hausdorff and of finite dimension, and if A has no simple quotient of type I, then A has the global Glimm halving property, see Definition 2.6 and [11]. A combination of this result with property (i) of Definition 1.2 gives a reduction to the case $m = 1$. Thus, if the primitive ideal space of A is a finite dimensional Hausdorff space and if A has no non-zero character, then property (i) of Definition 1.2 implies that A is purely infinite. Then we show that s.p.i. is implied by p.i. if the primitive ideal space is Hausdorff.

Summing up we get that for all C^*-algebras with Hausdorff primitive ideal space of finite dimension, for all C^*-algebras of real rank zero, and for all approximately divisible C^*-algebras the weakest definition of pure infiniteness (l.p.i.) implies the strongest one (s.p.i.). Moreover, in the case of C^*-algebras with infinite dimensional Hausdorff primitive ideal spaces pi(1) implies s.p.i.

Using the main result of [45], we deduce for the purely infinite separable stable nuclear C^*-algebras A with Hausdorff primitive ideal space the tensorial absorption property that $A \otimes \mathcal{O}_\infty \cong A$.

If we restrict our results to separable, stable and nuclear C^*-algebras A, then we can list our results in the following theorem:

Theorem 1.5. *Suppose that A is a separable, stable and nuclear C^*-algebra with Hausdorff primitive ideal space Prim(A). Then the following properties (i), (ii) and (iii) are equivalent:*

(i) *A is purely infinite(=pi(1)).*
(ii) *A is strongly purely infinite.*
(iii) *$A \otimes \mathcal{O}_\infty$ is isomorphic to A.*

If, moreover, Prim(A) is of finite dimension, then (i)–(iii) are equivalent to each of the following properties (iv)–(vii):

(iv) *A is weakly purely infinite.*
(v) *A is locally purely infinite.*
(vi) *Every simple quotient of A is purely infinite.*
(vii) *Every simple quotient B of A absorbs a copy of $\mathcal{O}_\infty$, i.e., $B \otimes \mathcal{O}_\infty \cong B$.*

It results now from [38], [39] that A as in Theorem 1.5 is classified up to isomorphisms by its $\mathcal{R}KK^G(\mathrm{Prim}(A), ., .)$-equivalence class (for trivial G).

The needed basic ingredients $h_0\colon A \to B$ of the theory in [38] can be constructed simply as follows: suppose that B is also as in Theorem 1.5 and that $X \cong \mathrm{Prim}(A) \cong \mathrm{Prim}(B)$. We show in section 5 that there is a non-degenerate $C_0(X)$-module and C^*-morphism from $C_0(X, \mathcal{O}_2 \otimes \mathcal{K})$ into B. A non-degenerate $C_0(X)$-module and *-monomorphism h_0 from A into B which represents the zero of $\mathcal{R}KK^G(X, A, B)$ can be defined as the composition $h_0 := \psi\varphi$ of ψ with a non-degenerate sub-trivialization $\varphi\colon A \hookrightarrow C_0(X, \mathcal{O}_2 \otimes \mathcal{K})$, see [10] for the existence of φ.

Note that Theorem 1.5 and [38] imply that for every separable nuclear C^*-algebra A with Hausdorff primitive ideal space Prim(A) there is a natural isomorphism

$$A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong C_0(\mathrm{Prim}(A), \mathcal{O}_2 \otimes \mathcal{K}).$$

In general one has the implications s.p.i. $\Rightarrow$ p.i. $\Rightarrow$ w.p.i., cf. [44], [45]. We show in section 4 that w.p.i. implies l.p.i. If the lattice of closed ideals of a C^*-algebra A is linearly ordered then A is l.p.i. if and only if it is purely infinite.

But following questions are open: does l.p.i. (respectively w.p.i., respectively p.i.) imply w.p.i. (respectively p.i., respectively s.p.i.) in general?

Suppose that A is a unital C^*-algebra with primitive ideal space Prim(A) isomorphic to $[0,1]^\infty$ and simple quotients isomorphic to $\mathcal{O}_2$. Is A purely infinite ? This question is also open if we assume moreover that A is pi(2).

We have the feeling that this question is related to the observation that there are non-stable separable C^*-algebras with the Hilbert cube $[0,1]^\infty$ as primitive ideal space and with simple quotients isomorphic to the compact operators on $\ell_2(\mathbb{N})$, cf. [11].

Let us close this introduction with a look to von Neumann algebras or, more generally, to AW*-algebras A, where we study A as a C^*-algebra. Then A has real rank zero as a C^*-algebra. Thus, A is locally purely infinite if and only if A is strongly purely infinite by Theorem 4.17. It follows from the logical sum of [45, cor. 6.9], [44, prop. 4.7] and [44, thm. 4.16] that a C^*-algebra A of real rank zero is strongly purely infinite if and only if every non-zero projection in A is properly infinite. This implies that an AW*-algebra A is of type III if and only if A satisfies one of our definitions of pure infiniteness.

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2. Preliminaries

We recall in this section a few basic results of the theory of (not necessarily locally trivial) continuous fields of C^*-algebras, on the projectivity of $C_0((0,1], M_n)$ and on the semi-projectivity of $\mathcal{O}_2$.

2.1. C(X)-algebras and C^*-bundles.

Let Y be a, not necessarily separated, topological space, for example the space of primitive ideals or of prime ideals of a C^*-algebra. By $C_b(Y)$ we denote the C^*-algebra of bounded continuous functions on Y with values in the complex numbers $\mathbb{C}$. Given a Hausdorff locally compact space X, let $C_0(X)$ denote the C^*-algebra of continuous functions on X with values in $\mathbb{C}$ and which vanish at infinity. Then we naturally identify $C_b(X)$ with the multiplier C^*-algebra of $C_0(X)$.

Definition 2.1. ([33]) A $C(X)$-*algebra* is a C^*-algebra A endowed with a non-degenerate *-morphism from $C_0(X)$ in the center of the multiplier C^*-algebra $\mathcal{M}(A)$ of A.

Here "non-degenerate" means that $C_0(X)A$ is dense in A. Thus A is nothing else but a quotient of $C_0(X, A)$ by a closed ideal, the $C_0(X)$-module structure is defined by this epimorphism, and the *-morphism from $C_0(X)$ to the center of $\mathcal{M}(A)$ extends uniquely to a unital strictly continuous *-morphism from $C_b(X)$ into the center of $\mathcal{M}(A)$.

Cohen factorization (cf. [8, prop. 1.8]), or the description of a $C(X)$-algebra as quotient of $C_0(X, A)$ shows that the set of products $C_0(\Omega)A = \{fa;\ f \in C_0(\Omega), a \in A\}$ is a closed ideal of A if Ω is an open subset of X. In particular $A = C_0(X)A$.

If $F \subset X$ is a closed subset we denote by $A_{|F}$ the quotient of A by the *closed* ideal $C_0(X \setminus F)A$. Note that $C_0(\Omega)A$ is also a $C(\Omega)$-algebra if Ω is an open subset of X

and that $A_{|F}$ is also a $C(F)$-algebra, because $C_0(X \setminus F) \subset C_0(X)$ is the kernel of the restriction map $C_0(X) \to C_0(F)$ and $C_0(X \setminus F)$ is contained in the kernel of the quotient-action of $C_0(X)$ on $A_{|F}$.

If $y \in X$ is a point of the Hausdorff space X, we write A_y for $A_{|\{y\}}$, and call A_y the *fiber* of A at $y \in X$.

Given an element $a \in A$, let a_y be the image of a in the fiber A_y ($y \in X$).

It holds $(fa)_y = f(y)a_y$ for $f \in C_b(X)$, $a \in A$ and $y \in X$, because $(f - f(y))C_0(X) \subset C_0(X \setminus \{y\})$ and $C_0(X)A = A$.
Thus, the function $N(a)\colon\ y \mapsto \|a_y\| := \|a + C_0(X \setminus \{y\})A\|$ satisfies

$$N(fa) = |f|N(a) \tag{2.1}$$

for $a \in A$ and $f \in C_b(X)$. In the same way one gets the the following formula of M. Rieffel [53] for $N(a)$:

$$N(a)\colon\ y \in X \mapsto \|a_y\| = \inf\{\|[1 - f(y)]a + fa\|\,;\ f \in C_0(X)_{sa}\}. \tag{2.2}$$

It is always upper semi-continuous, because the function $y \in X \mapsto \|[1 - f(y)]a + fa\|$ is continuous for fixed $f \in C_0(X)_{sa}$.

Let $\mathrm{Prim}(A)$ denote the primitive ideal space of A, cf. example 2.2.2. Since, by the Dauns–Hofmann theorem [51, cor. 4.4.8], there is a natural isomorphism from $C_b(\mathrm{Prim}(A))$ onto the center of $\mathcal{M}(A)$, we can equivalently define a $C(X)$-algebra A by a continuous map η from $\mathrm{Prim}(A)$ into the Stone-Čech compactification βX of X such that $\|fa + J\| = |f(\eta(J))|\,.\|a + J\|$ for $f \in C(\beta X)$, $a \in A$ and every primitive ideal J of A . The non-degeneracy condition $A = C_0(X)A$ in our definition implies that $\eta(\mathrm{Prim}(A)) \subset X$. Thus, $\|a\| = \sup\{N(a)(y);\ y \in X\}$, and for $x \in \mathrm{Im}(\eta)$ (and with the convention $\sup \emptyset := 0$),

$$\|a_x\| = \sup\{\|a + J\|\,;\ J \in \mathrm{Prim}(A),\ \eta(J) = x\}. \tag{2.3}$$

Definition 2.2. We say that the $C(X)$-algebra A is a *C^*-bundle* over X if the function $N(a)$ is moreover continuous for every $a \in A$ ([46], [8]).

Sometimes we write *continuous* C^*-bundle if we want to underline that the functions $N(a)$ are continuous.

Since we have assumed that $C_0(X)A$ is dense in A, we get $A = C_0(X)_+.A$ and then from (2.1) and $C_0(X)_+^2 = C_0(X)_+$ that $N(a)$ is even in $C_0(X)_+$ for $a \in A$. The reader can see from equations (2.2) and (2.3) for $N(a)(x)$, that a $C(X)$-algebra A is a C^*-bundle if and only if the above introduced continuous map $\eta\colon\ \mathrm{Prim}(A) \to X$ is also open (relatively to its image).

It is well-known that A is a C^*-bundle over X if and only if A is the C^*-algebra of continuous sections vanishing at infinity of a continuous field of C^*-algebras over X in the sense of [21, def. 10.3.1], such that the fibers are the A_x and that the *-morphism from $C_0(X)$ into $\mathcal{M}(A)$ coincides with the multiplication of continuous sections with functions, [8] and the discussion in [11].

(For the difference between continuous fields and our definition of C^*-bundles, let us consider $A := C_0(\mathbb{R})$ as C^*-bundle over the space $\mathbb{R}$. Then $A_t = \mathbb{C}$ for $t \in \mathbb{R}$ and the corresponding continuous field is $\mathcal{E} = ((A_t)_{t\in\mathbb{R}}, C(\mathbb{R}))$ where the unbounded continuous functions in $C(\mathbb{R})$ are considered as elements of the set $\prod_{t\in\mathbb{R}} A_t$.)

Remark 2.3. The elements of a C^*-bundle satisfy the following pull-back condition: *Let F and G be closed subsets of X.*

The natural epimorphism $A_{|(F\cup G)} \to A_{|F}$ and $A_{|(F\cup G)} \to A_{|G}$ defines $A_{|(F\cup G)}$ as the pullback of the epimorphism $A_{|F} \to A_{|(F\cap G)}$ and $A_{|G} \to A_{|(F\cap G)}$.

2.2. **Examples of C^*-bundles.**

2.2.1. If C is a $C(X)$-algebra and D is a C^*-algebra, the spatial tensor product $B = C \otimes D$ is endowed with a structure of $C(X)$-algebra through the map $f \in C_0(X) \mapsto f \otimes 1_{\mathcal{M}(D)} \in \mathcal{M}(C \otimes D)$. This $C(X)$-algebra is in general not a C^*-bundle over X.

If $C = C_0(X)$, the tensor product $B = C_0(X) \otimes D \simeq C_0(X; D)$ is a "trivial" C^*-bundle over X with constant fiber $B_x \cong D$. Thus, if $A \subset B$ is a closed $C_0(X)$-submodule and A is a C^*-subalgebra of B then A is a C^*-bundle over X.

Let A be a separable C^*-bundle over X with exact fibers A_x. If $\mathcal{O}_2$ is the unital Cuntz algebra generated by two isometries s_1, s_2 satisfying the relation $1_{\mathcal{O}_2} = s_1s_1^*+s_2s_2^*$ ([14]), then there exists a $C(X)$-linear *-monomorphism $A \hookrightarrow C(X)\otimes\mathcal{O}_2$ if and only if A is itself exact as a C^*-algebra, and this happens if and only if for every C^*-algebra D the $C(X)$-algebra $A \otimes D$ is again a C^*-bundle over X ([10, thm. A.1] and [46]). There exists a separable continuous C^*-bundle A over $\{0\} \cup \{1/n\,;\, n \in \mathbb{N}\} \subset [0, 1]$ with exact fibers such that A is not exact, [46].

2.2.2. We denote the *primitive ideal space* of a C^*-algebra A by $\mathrm{Prim}(A)$ The primitive ideals are kernels of irreducible representations of A . It is a T_0-space for the Jacobson topology (kernel-hull topology). A base of this topology is given by open sets of the form $\{K \in \mathrm{Prim}(A)\,;\, \|a+K\| > 0\}$ for some $a \in A_+$. Since $\|(a-t)_+ +K\| = (\|a+K\|-t)_+$ for $t > 0$ and $a \in A_+$, this means that the Jacobson topology is the coarsest topology on $\mathrm{Prim}(A)$ such that for every $a \in A$ the function $K \in \mathrm{Prim}(A) \mapsto \|a + K\|$ is lower semi-continuous.

On the other hand, for $a \in A$ and $t > 0$, the G_δ-subset $\{K \in \mathrm{Prim}(A)\,;\, \|a+K\| \geq t\}$ of $\mathrm{Prim}(A)$ is quasi-compact, [21, prop. 3.3.7].

If the space $\mathrm{Prim}(A)$ is in addition Hausdorff, then this yields that $\mathrm{Prim}(A)$ is locally compact and that the functions $N(a)\colon\ K \in \mathrm{Prim}(A) \mapsto \|a + K\|$ are continuous functions on $\mathrm{Prim}(A)$ which vanish at infinity, [21, cor. 3.3.9]. Then the Dauns–Hofmann theorem [51, cor. 4.4.8] implies that A is naturally a C^*-bundle over $\mathrm{Prim}(A)$ with simple fiber $A_K = A/K$ at $K \in \mathrm{Prim}(A)$.

2.3. **Projectivity of $M_n(C_0(0, 1])$.** (See also [48], [49].)

Let $\{e_{i,j}\}_{i,j\in\mathbb{N}}$ denote the canonical system of matrix units of the C^*-algebra $\mathcal{K} := \mathcal{K}(\ell_2(\mathbb{N}))$ of compact operators acting on the separable infinite dimensional Hilbert space $\ell_2(\mathbb{N})$. These operators satisfy the relations $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$ and $e_{i,j}^* = e_{j,i}$.

As the function $h_0 : t \in (0, 1] \mapsto t \in \mathbb{C}$ generates $C_0((0, 1])$, one gets that for $n > 1$, $C_0((0, 1])\otimes M_n(\mathbb{C})$ is the universal C^*-algebra generated by $n-1$ contractions $f_2, \ldots , f_n$ satisfying the relations

$$f_if_j = 0 \quad \text{and} \quad f_i^*f_j = \delta_{i,j}\; f_2^*f_2 \quad \text{for} \quad 2 \leq i, j \leq n\,. \tag{2.4}$$

The natural C^*-algebra epimorphism Φ from $C_0((0,1])\otimes M_n(\mathbb{C})$ onto $C^*(f_2,\dots,f_n)$ is uniquely determined by

$$\Psi\colon h_0\otimes e_{j,1}\mapsto f_j \qquad \text{for } 1<j\le n.$$

Note that $f_j := g_j(g_1)^*$ $(1<j\le n)$ satisfy (2.4) if $g_1,\dots,g_n$ just satisfy $g_i^*g_j=\delta_{i,j}\, g_1^*g_1$.

Moreover, *the C^*-algebra $C_0((0,1],M_n(\mathbb{C}))$ is projective, i.e., for every closed ideal $J\subset A$ and every *-homomorphism of C^*-algebras $\psi\colon C_0((0,1],M_n(\mathbb{C}))\to A/J$ there is a *-homomorphism $\varphi\colon C_0((0,1],M_n(\mathbb{C}))\to A$ with $\pi_J\varphi=\psi$.* (cf. [48, thm. 10.2.1], [49] for other proofs and equivalent definitions).

Proof. Let $b_k=\psi(h_0\otimes e_{k,1})\in A/J$, and choose a selfadjoint contraction $c\in A$ satisfying

$$c+J=(b_2)^*b_2-b_{n+1}(b_{n+1})^*+(\sum_{k=2}^{n} b_k(b_k)^*)\,.$$

There is a contraction $a\in A$ with $a+J=b_{n+1}(b_{n+1}^*b_{n+1})^{-1/3}$ in A/J.

By induction assumption there must be $g_2,\dots,g_n\in\overline{c_+Ac_+}$ with $g_k+J=b_k$, $g_ig_k=0$ and $g_i^*g_k=\delta_{i,k}g_2^*g_2$. (The condition is void if $n=1$.)

Now define $f\in A$ by $f:=(c_-)a(g_2^*g_2)^{1/3}$ and consider the polar decompositions $v_k(g_2^*g_2)^{1/2}$ of g_k in A^{**}. Then v_k is in $pA^{**}p$, where p is the support projection of c_+ in A^{**}. Thus, $f^2=0$, $v_kf=v_k^*f=0$, the partial isometries v_k satisfy $v_iv_k=0$, $v_i^*v_k=\delta_{ik}v_2^*v_2$ and $v_2^*v_2$ is the support projection of $g_2^*g_2\ge f^*f$.

Since f^*f is in $\overline{g_2^*Ag_2}=\overline{g_k^*Ag_k}$, we get that

$$f_k:=v_k(f^*f)^{1/2}=\lim_{n\to\infty} g_k(g_k^*g_k)^{1/n-1/2}(f^*f)^{1/2}$$

exists, is in A and $f_k+J=b_k$ in A/J for $k=2,\dots,n$. Then $f_1,\dots,f_n$ and $f_{n+1}:=f$ satisfy the defining relations for a *-homomorphism $\varphi\colon C_0((0,1],M_n(\mathbb{C}))\to A$ with $\varphi(h_0\otimes e_{k,1})=f_k$, and $\pi_J\varphi=\psi$. $\square$

The advantage of the projectivity of $C_0((0,1],M_n(\mathbb{C}))$ is the following refined version of the Glimm halving lemma:

If $d\colon B\to\mathcal{L}(\mathcal{H})$ is an irreducible representation of a C^*-algebra B of dimension $\ge n$ and if $p\in\mathcal{L}(\mathcal{H})$ is an orthogonal projection onto an n-dimensional subspace, then we can define $A:=\{b\in B;\, pd(b)=d(b)p\}$ and $J:=\{b\in A;\, pd(b)p=0\}$. The restriction of d to A defines an *isomorphism* from A/J onto $p\mathcal{L}(\mathcal{H})p\cong M_n(\mathbb{C})$ by a slight sharpening of the Kadison transitivity theorem, cf. [7, prop. 3.4] or [36, thm. 1.4(iii)] or in the unital case [51, 2.7.5 and 3.11.9].

Thus, *there is a morphism $\varphi\colon C_0((0,1],M_n(\mathbb{C}))\to A\subset B$ such that $a\mapsto d(\varphi(a))p$ is a *-epimorphism onto $p\mathcal{L}(\mathcal{H})p$ with kernel $C_0((0,1),M_n(\mathbb{C}))$.*

2.4. **On semi-projectivity.** More generally a separable C^*-algebra B is said to be *semi-projective* ([3]) if for any C^*-algebra A, any increasing sequence $\{J_k\}$ of (closed two-sided) ideals in A and any *-morphism $\varphi\colon B\to A/J_\infty$, where $J_\infty=\overline{\cup J_k}$, there exists an index n and a *-morphism ψ from B to A/J_n such that $\varphi=\pi_n\circ\psi$, where $\pi_n\colon A/J_n\to A/J_\infty$ is the natural quotient map.

The C^*-algebras $\mathbb{C}$, $\mathbb{C}\oplus\mathbb{C}$, $\mathcal{T} := C^*(s : s^*s = 1)$, $\mathcal{E}_2 := C^*(s_1, s_2 : s_i^* s_j = \delta_{ij}1)$, $\mathcal{O}_2$ are semi-projective, as the reader easily can check step by step (with help of the functional calculus), see exercises 4.7 (c)-(e) of [4]. On the other side semi-projectivity is not invariant under stabilization: $C_0(0,1]\otimes\mathcal{K}$ and $\mathcal{K}$ are not semi-projective. (We do not know whether $\mathcal{O}_2\otimes\mathcal{K}$ is semi-projective or not.) Moreover we have the following extension property.

*If D is a separable C^*-bundle over a locally compact Hausdorff space X , F is a compact subset of X and φ is a *-morphism from a semi-projective C^*-algebra B to $D_{|F}$, then there exist a compact subset G of X and a *-morphism $\psi\colon B \to D_{|G}$ such that F is contained in the interior of G and $\psi(a)_{|F} = \varphi(a)$ for all $a \in B$.*

Moreover, if D has simple fibers and if B is unital and $\varphi(1)$ generates $D_{|F}$ as a closed ideal, then G can be found such that also $\psi(1)$ generates $D_{|G}$ as closed ideal.

Proof. Since B is separable we find a separable C^*-subalgebra C of D such that $\varphi(B)$ is contained in the image $\pi_F(C)$ of C under the canonical epimorphism $\pi_F\colon B \to B_{|F}$.

For every compact neighborhood G of F we let $I_G := C\cap C_0(X\setminus G)_+D$. The definition of neighborhoods implies that F is contained in the interior of G. The I_G define an upward directed family of closed ideals of C with closed union equal to $C\cap C_0(X\setminus F)D$. Since C is separable, there exists a countable sequence $G_1 \subset G_2 \subset \ldots$ such that the closure of the union of the I_{G_n} is the same as the closure of the union of the I_G.

The semi-projectivity of B implies that there is $n \in \mathbb{N}$ and a *-homomorphism $\psi_n\colon B \to C/I_{G_n} \subset D_{|G_n}$ with $\pi_{G_n,F}\psi_n = \varphi$, where $\pi_{G,F}(d) = d_{|F}$ for $d \in D_{|G}$, $F \subset G$. Take $G := G_n$ and $\psi := \psi_n$ if B is non-unital.

If B is unital, then $N(\psi_n(1))(x) = N(\varphi(1))(x) > 0$ for $x \in F$ and $N(\psi_n(1))$ takes only the values 0 and 1. Thus, its support G is a compact and (relatively to G_n) open subset of G_n. Therefore, G must contain F in its interior (relative to X), and $a \in B \mapsto \psi(a) := \psi_n(a)_{|G}$ defines a *-homomorphism from B into $D_{|G}$, such that $\psi(1)$ generates $D_{|G}$ as a closed ideal and $\psi(a)_{|F} = \varphi(a)$ for $a \in B$. $\square$

2.5. **Finite dimensional Hausdorff spaces.**

Recall that a compact Hausdorff space X has *(covering-) dimension* $dim(X) \leq n \in \mathbb{N}$ if for every finite open covering of X there is another covering of X by open subsets which refines the given covering and is such that the intersection of every $n+2$ distinct sets of this covering is always empty, i.e., a given finite open covering admits a refinement whose nerve is a simplicial complex of dimension $\leq n$.

Definition 2.4. We say that a topological space X has the *decomposition-dimension* $\leq m$ if for every finite covering $\mathcal{O}$ of the topological space X, there is a finite open covering $\mathcal{U} = \{U_1, \ldots, U_q\}$ which refines $\mathcal{O}$ and for which there exists a map $\iota\colon \{1, \ldots, q\} \to \{1, \ldots, m+1\}$, such that for each $1 \leq k \leq m+1$, the open set $Z_k = \bigcup_{j\in\iota^{-1}(k)} U_j$ is the *disjoint* union of the open sets U_j, $j \in \iota^{-1}(k)$.

Later we use the following lemma of [11].

Lemma 2.5 (([11])). *Let X be a compact Hausdorff space of topological dimension $\leq n$, let $\mathcal{O} = \{O_1, \ldots, O_p\}$ be an open covering of X and let $\mathcal{U} = \{U_1, \ldots, U_q\}$ be an* *open covering of X which is an refinement of $\mathcal{O}$ such that every intersection of $n+2$ different elements of $\mathcal{U}$ is empty.*

Then there is a finite open covering $\mathcal{V}$ of X which is a refinement of $\mathcal{U}$ (and thus of $\mathcal{O}$) and is such that the set $\mathcal{V}$ can be partitioned into $n+1$ subsets, consisting of elements with pairwise disjoint closures.

The lemma says that a compact Hausdorff space X has covering-dimension $\leq n$ if and only if it has decomposition-dimension $\leq n$. It is not known if this also holds for T_0-spaces like $\mathrm{Prim}(A)$.

2.6. Global Glimm halving for C^*-bundles.

In [11] the authors have studied a global version of the Glimm halving for non-simple C^*-algebras (Definition 2.6). There is proven that this global property holds for C^*-algebras with Hausdorff finite dimensional primitive ideal space and with no type I quotients (Theorem 2.7).

Glimm lemma (cf. [51, lemma 6.7.1], [60, lemma 4.6.6], or subsection 2.3) can be equivalently restated as follows: given any non-zero positive element a in a C^*-algebra A such that $\overline{aAa}$ is not a commutative algebra, there exists a non-zero element $b \in \overline{aAa}$ with $b^2 = 0$. This property motivates the following definition.

Definition 2.6. A C^*-algebra A is said to have the *global Glimm halving property* if for every positive $a \in A_+$ and every $\varepsilon > 0$, there exists $b \in \overline{aAa}$ such that $b^2 = 0$ and $(a - \varepsilon)_+$ belongs to the closed ideal $\overline{AbA}$ generated by b .

The global Glimm halving property of a C^*-algebra A implies by induction that for all $a \in A_+$, $\varepsilon > 0$ and $n \geq 2$, there exists a *-homomorphism $\pi_n \colon C_0((0,1]) \otimes M_n(\mathbb{C}) \to \overline{aAa}$ such that $(a - \varepsilon)_+$ is in the ideal generated by the image of π_n (cf. [11]). In particular A can not have any irreducible representation which contains the compact operators in its image, hence A *is strictly anti-liminal*, i.e., every non-zero quotient of A is anti-liminal.

Theorem 2.7. *(*[11]*) Let A be a continuous C^*-bundle over a finite dimensional locally compact Hausdorff space X and suppose that each fiber A_x is simple and not of type I .*

Then the global Glimm halving property 2.6 holds for A .

Remark 2.8. In [11] it is shown: If $B_1, B_2, \ldots$ is a sequence of simple unital C^*-algebras $\neq \mathbb{C}$, then $A \otimes B_1 \otimes B_2 \otimes \ldots$ satisfies the global Glimm halving property. Strictly anti-liminal AF-algebras have global Glimm halving property.

2.7. Majorization and properly infinite elements.

The positive and negative parts of a selfadjoint element $a \in A$ are denoted

$$a_+ := (|a| + a)/2 \in A_+ \quad \text{and} \quad a_- := (|a| - a)/2 \in A_+ \,.$$

Suppose now that $a, b \in A_+$ and $\varepsilon > 0$ verify $\|a - b\| < \varepsilon$. Then the positive part $(b - \varepsilon)_+ \in A$ of $(b - \varepsilon.1) \in \mathcal{M}(A)$ admits the decomposition $(b - \varepsilon)_+ = d^*ad$ for some contraction $d \in A$ ([45, lemma 2.2]). Thus, if $a_1, a_2, \ldots$ is a sequence of positive elements in A_+ converging to a or, more generally, satisfying $\limsup \|b - a_n\| < \varepsilon$, then for $n \in \mathbb{N}$ large enough, there are contractions $d_n \in A$ such that

$$(b - \varepsilon)_+ = d_n^* a_n d_n \,. \tag{2.5}$$

In particular, if $\eta > 0$ is small enough, there exists a contraction d_η in A with $(b - \varepsilon)_+ = d_\eta^*(a - \eta)_+ d_\eta$.

We derive two other consequences of [45, lemma 2.2]:

(i) *If $\delta \in [0, \infty)$ and $0 \le b \le a + \delta.1$ (in $\mathcal{M}(A)$), then for every $\varepsilon > \delta$ there is a contraction $f \in A$ such that $(b - \varepsilon)_+ = f^* a f$*

(ii) *If $c, d \in A_+$ and d is in the closed ideal generated by c, then for every $\varepsilon \in (0, 1)$ there are $p \in \mathbb{N}$, $e_1, \ldots, e_p \in A$ and $\eta > 0$ such that $(d - \varepsilon)_+ = \sum e_j^*(c - \eta)_+ e_j$.*

Proof. (i) If $e_n := (a + \delta + 1/n)^{-1/2} b^{1/2}$, $c_n := (a + \delta + 1/n)^{-1}(a + \delta)$ and $a_n := e_n^* a e_n$, then $\|c_n\| < 1$, $e_n e_n^* \le c_n$ and $b - a_n = (\delta + 1/n) e_n^* e_n$, which implies $\|e_n\| < 1$ and thus $\limsup \|b - a_n\| \le \delta < \varepsilon$. Let $f := e_n d_n$ for sufficiently large $n \in N$.

(ii) We may suppose that $(d - \varepsilon)_+ \neq 0$, i.e., $\varepsilon < \|d\|$. The element $d^{1/2}$ is in the closed linear span of $Ac^{1/2}A$. Let $\delta := \varepsilon/(4\|d\|^{1/2} + 1)$ and $\gamma := (\delta + 2\|d\|^{1/2})\delta$. Then $\varepsilon - \gamma > 0$, $\delta < \|d\|^{1/2}$ and there are $p \in \mathbb{N}$ and non-zero columns $f, g \in M_{p,1}(A)$ with $\|d^{1/2} - v\| < \delta$, where $v := g^*(c^{1/2} \otimes 1_p) f$. Let $\eta > 0$ with $\eta(\|g\|\|f\|)^2 < \varepsilon - \gamma$. Straightforward calculations show $\|d - v^* v\| < \gamma$ and

$$v^* v \le \|g\|^2 (f^*((c - \eta)_+ \otimes 1_p) f) + \varepsilon - \gamma \,.$$

Part (i) gives a contraction $h \in A$ with

$$(d - \gamma)_+ = h^* v^* v h \le \|g\|^2 (fh)^*((c - \eta)_+ \otimes 1_p)(fh) + \varepsilon - \gamma$$

and then $e \in M_{p,1}(A)$ with $(d - \varepsilon)_+ = e^*((c - \eta)_+ \otimes 1_p) e$. □

Remarks 2.9. (i) If $a \in A_+$, $b \in M_n(A)_+$ and there is a matrix $e \in M_{m,n}(A)$ with $\|b - e^*(a \otimes 1_m) e\| < \varepsilon$ for a constant $\varepsilon > 0$, then

$$(b - \varepsilon)_+ = f^*((a - 2\eta)_+ \otimes 1_m) f$$

for some matrix $f \in M_{m,n}(A)$ with $\|f\| \le \|e\|$ and some $\eta \in (0, \varepsilon)$: indeed we find $\eta > 0$ such that we still have $\|b - e^*((a - 2\eta)_+ \otimes 1_m) e\| < \varepsilon$. As shown above there is a contraction $d \in M_n(A)$ such that $f := ed$ is as desired.

(ii) A non-zero positive element $a \in A_+$ in a (not necessarily purely infinite) C^*-algebra A is *properly infinite* if, for every $\varepsilon > 0$, there exists a row matrix $d = (d_1, d_2) \in M_{1,2}(A)$ such that $\|d^* a d - a \otimes 1_2\| < \varepsilon$, cf. [44, def. 3.2].

If one applies (i) with $m = 1, n = 2$ then one finds $u, v \in aAa$ with $u^* u = v^* v = (a - \varepsilon)_+$ and $u^* v = 0$ ([44, prop. 3.3]), i.e., there exits a row $w = (u, v) \in M_{1,2}(aAa)$ satisfying

$$w^* w = (a - \varepsilon)_+ \otimes 1_2 \quad \text{in } \; A \otimes M_2(\mathbb{C}) \,.$$

An element $a \in A_+$ is properly infinite if for every closed ideal J of A which does not contain a there is an element $h \neq 0$ in $(A/J)_+$ such that for every $\delta > 0$ there exists a row matrix $d = (d_1, d_2) \in M_{1,2}(A/J)$ with $\|d^* \pi_J(a) d - (\pi_J(a) \oplus h)\| < \delta$, cf. [44, prop. 3.14].

(iii) A C^*-algebra A is purely infinite if and only if every element $a \in A_+ \setminus \{0\}$ is properly infinite, [44, thm. 4.16].

(iv) *Purely infinite C^*-algebras A have the global Glimm halving property 2.6:* namely, if $a \in A_+ \setminus \{0\}$ and $\varepsilon > 0$, then $b = vu^* \in aAa$ with u, v from (ii) verifies $b^2 = 0$ and $[(a-\varepsilon)_+]^2 = v^*bu$, so that $(a-\varepsilon)_+ \in AbA$.

Lemma 2.10. *Given a positive element a in a C^*-algebra A and $0 < \varepsilon < \|a\|$, suppose that for every $\nu > 0$, the element $(a-\nu)_+$ is either zero or properly infinite (cf. Remark 2.9(ii)).*

Then there exists an infinite sequence $w_1, w_2, \ldots$ in aAa such that for all $n, m \in \mathbb{N}$, one has $w_n^* w_m = \delta_{n,m}\,(a-\varepsilon)_+$.

The element $d = \sum_{n\in\mathbb{N}} 2^{-n} w_n w_n^*$ generates a stable hereditary C^*-subalgebra $\overline{dAd}$ of $\overline{aAa}$ such that $(a-\varepsilon)_+^2$ is in the ideal generated by d.

Proof. For $n \in \mathbb{N}$, let $\varepsilon_n := 2^{-n-1}.\varepsilon$ and $\delta_n := \sum_{0\le k\le n} \varepsilon_k = (1-2^{-n-1}).\varepsilon < \varepsilon$. From Remark 2.9(ii) one can see that a itself is properly infinite if $(a-\nu)_+$ is properly infinite for every $\nu \in (0,\delta)$ for some $\delta > 0$, and that b^*b and $(b^*b)^{1/2}$ are properly infinite if bb^* is properly infinite.

Thus, if we let $v_{-1} := a^{1/2}$, then we can find inductively (by repeated use of Remark 2.9(ii)) elements u_n, v_n in $v_{n-1}Av_{n-1}^*$ ($n \in \mathbb{N}$) such that

a) for every $\nu > 0$, the element $(v_n^* v_n - \nu)_+$ is either zero or properly infinite,
b) $u_n^* u_n = v_n^* v_n = (v_{n-1}^* v_{n-1} - \varepsilon_n)_+ = (a-\delta_n)_+ \ge (a-\varepsilon)_+$ and
c) $u_n^* v_n = 0$.

For $n \in \mathbb{N}$, let $\phi_n\colon \mathbb{R}_+ \to [0,1]$ be the function $\phi_n(t) = \begin{cases} 0 & \text{if } t \le \varepsilon, \\ (t-\varepsilon)/(t-\delta_n) & \text{if } t \ge \varepsilon\,. \end{cases}$
Then the elements $w_n = u_n \phi_n(a)^{1/2} \in (a-\varepsilon/2)_+ A (a-\varepsilon/2)_+$, $n \in \mathbb{N}$, satisfy the requested relations. □

2.8. **Prime ideals of tensor products.**

We consider here the T_0-space prime$(A \otimes B)$ of prime ideals of the spatial tensor product $A \otimes B$ of C^*-algebras A and B. The structure of this space is important for the question when tensor products $A \otimes B$ are locally purely infinite. (The second named author shows in [40] that the example of Rørdam [59] allows to construct an example of a strongly purely infinite C^*-algebra B such that $A \otimes B$ is not locally purely infinite for a certain C^*-algebra A.)

Remarks 2.11. Recall that a closed ideal $I \neq A$ of a C^*-algebra A is *prime* if $J \cap K \subset I$ implies $J \subset I$ or $K \subset I$ for closed ideals $J, K \triangleleft A$. Since $JK = J \cap K$, this says equivalently that $aAb \subset I$ implies $a \in I$ or $b \in I$.

Kernels of factorial representations are prime. The hull-kernel topology makes the set of prime ideals $I \neq A$ of A to a T_0-space prime(A) which contains the primitive ideal space Prim(A) as a dense subspace. Conversely, prime(A) is naturally isomorphic to the T_0-space of prime closed subsets of Prim(A). Thus, Prim$(A) = $ prime(A) if Prim(A) or prime(A) is Hausdorff.

As in the case of the primitive ideal space there is a one-to-one correspondence between open subsets Z_J of prime(A) and closed ideals of J of A given by

$$J \mapsto Z_J := \{I \in \text{prime}(A) \colon J \not\subset I\}.$$

We say that A is *prime* if 0 is a prime ideal of A. Clearly $J \triangleleft A$ is prime if and only if A/J is prime.

It is easy to see that the Hamana envelope and the Dedekind AW*-completion of a prime C^*-algebra A are AW*-factors (which are always primitive but not simple). Thus, prime ideals of C^*-algebras are the kernels of *-homomorphisms into AW*-factors with images which are dense in a certain AW*-sense.

A result of Dixmier ([20]) says: if A is separable then prime(A) and Prim(A) are the same, cf. [51, prop.4.3.6].

Nik Weaver gave in 2001 an example ([63]) of a non-separable prime C^*-algebra which is not primitive.

Lemma 2.12. (i) *If N is a C^*-seminorm on the algebraic tensor product $A \odot B$ with $N(a \otimes b) \neq 0$ for $a \otimes b \neq 0$, then N majorizes the spatial C^*-norm on $A \odot B$.*

(ii) *Every non-zero closed ideal $I \triangleleft A \otimes B$ contains a non-zero elementary tensor $a \otimes b$.*

(iii) *Suppose that $J \triangleleft A$ and $K \triangleleft B$ are closed ideals. Let I denote the kernel of the epimorphism $A \otimes B \to (A/J) \otimes (B/K)$. Then the closure I_0 of the sum of ideals generated by elementary tensors $a \otimes b \in I$ is $I_0 = J \otimes B + A \otimes K$.*

If $I_0 = I$ then the kernel of $(A/J) \otimes B \to (A/J) \otimes (B/K)$ is $(A/J) \otimes K$.

(iv) *If $J_1 \subset K_1 \triangleleft A$ and $J_2 \subset K_2 \triangleleft B$, then $J_1 \otimes B + A \otimes J_2 = K_1 \otimes B + A \otimes K_2 \neq A \otimes B$ implies $J_1 = K_1$ and $J_2 = K_2$.*

Proof. (i): It is easy to check that the restrictions of N to $C \odot B$ and $A \odot D$ for commutative C^*-subalgebras $C \subset A$ and $D \subset B$ are the (unique) C^*-norm there (i.e., check the special case of (ii) for commutative A or B). But this is the only requirement needed in the proof of Takesaki in [61] that for every pure state φ of A the set of pure states ψ of B with $|(\varphi \otimes \psi)(d)| \leq N(d)$ for $d \in A \odot B$ is separating for B. The latter implies that N majorizes the spatial C^*-norm.

(ii): If I does not contain a non-zero elementary tensor, then the C^*-seminorm N on $A \odot B$ which defined by the *-homomorphism $A \odot B \to (A \otimes B)/I$ satisfies the assumption of (i).

(iii): The kernel of $A \odot B \to (A/J) \odot (B/K) \subset (A/J) \otimes (B/J)$ is equal to $J \odot B + A \odot K$. Thus, the closed ideal of $A \otimes B$ which is generated by the elementary tensors in the kernel of $A \otimes B \to (A/J) \otimes (B/K)$ is equal to $J \otimes B + A \otimes K$.

(iv): Since $K_1 \otimes B + A \otimes K_2 \neq A \otimes B$, there are $c \in A$, $\varphi \in A^*$, $d \in B$, $\psi \in B^*$ with $\varphi(K_1) = 0$, $\varphi(c) = 1$, $\psi(K_2) = 0$ and $\psi(d) = 1$. Since $(\text{id}_A \otimes \psi)(A \odot J_2) = 0$ and $(\text{id}_A \otimes \psi)(J_1 \odot B) \subset J_1$, we have $a = (\text{id}_A \otimes \psi)(a \otimes d) \in J_1$ for $a \in K_1$, i.e., $K_1 = J_1$. $\square$

Lemma 2.13. *Let A, B be C^*-algebras and let I be a prime ideal of $A \otimes B$.*

(i) *The sets* $I_A := \{a \in A\colon a \otimes B \subset I\}$ *and* $I_B := \{b \in B\colon A \otimes b \subset I\}$ *are prime ideals of* A *and* B *respectively.*
In particular $J = I_A$ *and* $K = I_B$ *if, in addition,* $I = J \otimes B + A \otimes K$.

(ii) *If* $a \otimes b$ *is in* I *then* $a \in I_A$ *or* $b \in I_B$.

(iii) *The equality* $I_A \otimes B + A \otimes I_B = 0$ *implies* $I = 0$.

(iv) *The ideal* I *is contained in the kernel of* $A \otimes B \to (A/I_A) \otimes (B/I_B)$.

(v) *If* $J \lhd A$ *and* $K \lhd B$ *are prime, then the kernel* I_0 *of* $A \otimes B \to (A/J) \otimes (B/K)$ *is prime.*
If, moreover, $I_0 = A \otimes L + M \otimes B$ *for some closed ideals* $L \lhd A$ *and* $M \lhd B$ *then* $J = L = I_A$ *and* $K = M = I_B$.

Proof. (i): Clearly I_A is a closed ideal of A. If $J \lhd A$ and $K \lhd A$ are closed ideals such that $JK \subset I_A$ then $(J \otimes B)(K \otimes B) \subset I$. Thus, $J \otimes B \subset I$ or $K \otimes B \subset I$, which implies $J \subset I_A$ or $K \subset I_A$. Thus, I_A is prime. The same happens with I_B.

If $J \otimes B + A \otimes K \subset I$ then $J \subset I_A$ and $K \subset I_B$. Now apply (iv) of Lemma 2.12

(ii): Let J_1 and J_2 denote the closed ideals generated by a and b respectively. Then $(J_1 \otimes B)(A \otimes J_2) \subset I$.

(iii) follows from (ii) and part (ii) of Lemma 2.12.

(iv): The C^*-seminorm N on the algebraic tensor product $(A/I_A) \odot (B/I_B)$ which is given by the natural *-homomorphism from $(A/I_A) \odot (B/I_B)$ into $(A \otimes B)/I$ is non-zero on non-zero elementary tensors $(a + I_A) \otimes (b + I_B) = (a \otimes b) + (I_A \odot B) + (A \odot I_B)$ by (i) and (ii). Thus, part (i) of Lemma 2.12 applies and gives that N majorizes the spatial norm on $(A/I_A) \odot (B/I_B)$, which means that I is contained in the kernel of $A \otimes B \to (A/I_A) \otimes (B/I_B)$.

(v): By Remarks 2.11 we may assume that $J = 0$ and $K = 0$, i.e., we have to show that $A \otimes B$ is prime if A and B are prime. Suppose that P an Q are non-zero closed ideals of $A \otimes B$ such that $PQ = 0$. By part (ii) of Lemma 2.12, there are non-zero elements $a, c \in A$, $b, d \in B$ such that $a \otimes b \in P$ and $c \otimes d \in Q$. Thus, $(aec) \otimes (bfd) = 0$ for all $e \in A$, $f \in B$, which implies $aAc = 0$ or $bBd = 0$. This contradicts that A and B *both* are prime. □

Lemma 2.14. *Let* φ *be a pure state on a* C^**-algebra* A *and* $G \subset A$ *be a separable* C^**-subalgebra. Then there exist a separable* C^**-subalgebra* $B \subset A$ *and* $b \in B_+$ *with* $\|b\| = 1$ *such that* $G \subset B$, $\varphi(b) = 1$ *and* $\{a \in B;\, \varphi(a) = 0\} \subset \overline{(b - b^2)B} + \overline{B(b - b^2)}$. *The restriction* $\varphi|B$ *is pure,* $\{d \in B;\, \varphi(d^*d + dd^*) = 0\} = \overline{(b - b^2)B(b - b^2)}$ *and* $\lim_{n\to\infty} \|b^n d b^n - \varphi(d) b^{2n}\| = 0$ *for all* $d \in B$.

Proof. By a variant of Kadison's transitivity theorem (cf. end of subsection 2.3 with $n = 1$) we find $k \in A_+$ with $\|k\| = 1$ and $\varphi(k) = 1$. Then we have $\varphi(d) = \varphi(dk) = \varphi(kd)$ for all $d \in A$. The left ideal L of A defined by $L = \{a \in A;\, \varphi(a^*a) = 0\}$ is closed and $\ker(\varphi) = L^* + L$, because φ is pure (cf. [51, prop. 3.13.6]). Thus, $\{a - \varphi(a)k;\, a \in A\} \subset L^* + L$ and there exists a separable C^*-subalgebra B_1 of A with $G \subset B_1$ and $Z(G) \subset (L^* \cap B_1) + (L \cap B_1)$ where $Z(G) = \{a - \varphi(a)k;\, a \in G\}$. (Note here that $(L^* \cap B_1) + (L \cap B_1)$ is closed, $G \cap \ker(\varphi) \subset Z(G) \subset B_1$ and $k \in B_1$.)
If we repeat this construction with $B_1, B_2, \ldots$ in place of G we get a sequence of

separable C^*-subalgebras $G \subset B_1 \subset B_2 \subset \ldots A$ such that $k \in B_n$ and $B_n \cap \ker(\varphi) \subset L^* \cap B_{n+1} + L \cap B_{n+1}$. If B denotes the closure of $\bigcup_n B_n$, then $B \cap \ker(\varphi) = L^* \cap B + L \cap B$, because $B \cap \ker(\varphi)$ is the closure of $\bigcup_n (B_n \cap \ker(\varphi))$ and $B \cap \ker(\varphi)$ is the image of the bounded linear projection $Z\colon a \mapsto a - \varphi(a)k$ in $\mathcal{L}(B)$. In particular the restriction of φ to B is a pure state.

Let h be a strictly positive contraction in the separable C^*-algebra $L^* \cap B \cap L$. Then $h \in B_+$, $\varphi(h) = 0$ and $B \cap \ker(\varphi) = \overline{Bh} + \overline{hB}$. Thus, $h + k$ is a strictly positive element of B. If E is the separable C^*-subalgebra of B generated by h and k, then the restriction of φ to E is a character, because h and k are in the multiplicative domain of the completely positive map $\varphi\colon A \to \mathbb{C}$. It follows that $J := E \cap ker(\varphi)$ is an ideal with $h \in J \subset \overline{hBh}$. Let $f := \|2h\|^{-1}h$. Then $b := (1-f)^{1/2}(k+f)(1-f)^{1/2}$ is a strictly positive element in E with $\varphi(b) = 1$ and $\|b\| \leq \|b + f^2\| \leq 1$, because $\varphi(f) = 0$, $1/2 \leq (1-f)$ and $(1-f)^{1/2}k(1-f)^{1/2} \leq 1 - f$. We get that $b^{1/2}f^2b^{1/2} \leq b(1-b)$ and $\varphi(b(1-b)) = 0$, i.e., $b(1-b) \in J$. Since $b^{1/2}f^2b^{1/2}$ is a strictly positive element of J, it follows that $b(1-b) \geq b^{1/2}f^2b^{1/2}$ is a strictly positive element of J. Since $h \in J \subset \overline{hBh}$, we get that $b(1-b)$ is a strictly positive element of $\overline{hBh}$. Thus, b is a strictly positive element of B with $\varphi(b) = 1$, $\|b\| = 1$, $\overline{Bh} = \overline{Bb(1-b)}$ and $B \cap \ker(\varphi) = \overline{b(1-b)B} + \overline{Bb(1-b)}$. Since $\|b^n(1-b)\| < 1/n$ (by functional calculus), we get $\lim_{n\to\infty} \|b^n d b^n - \varphi(d)b^{2n}\| = 0$ for $d \in B$. □

Lemma 2.15. *If A and B are C^*-algebras and if D is a non-zero hereditary C^*-subalgebra of the minimal C^*-algebra tensor product $A \otimes B$, then there exists $0 \neq z \in A \otimes B$ with $zz^* \in D$ and $z^*z = e \otimes f$ for some non-zero $e \in A_+$ and $f \in B_+$.*

If $d \in D_+$ and $\varphi \in A^*$ and $\psi \in B^*$ are pure states with $(\varphi \otimes \psi)(d) > 0$, then z can be taken such that, moreover, $\varphi(e)\psi(f) > 0$.

Proof. Let $d \in D_+$ with $\|d\| = 1$, and let $C := A \otimes B$. The minimal C^*-algebra tensor product is the spatial tensor product w.r.t. the direct sum of irreducible representations (as follows e.g. from [60, prop. 1.22.9]). Thus, there are pure states φ on A and ψ on B such that $(\varphi \otimes \psi)(d) > 0$.

We assume from now on that we are given a fixed contraction $d \in D_+$ and fixed pure states φ and ψ with $(\varphi \otimes \psi)(d) > 0$ (to prove also the second part). Then $a := (\varphi \otimes \mathrm{id}_B)(d) \in B_+$ is a non-zero contraction and $0 < \psi(a) \leq \|a\|$.

Let $\delta := \psi(a)/2$ and $f := (a - \delta)_+^2$. Thus, $0 < \delta \leq 1/2$, $f \in B_+$ and $\psi(f) > 0$, because $\psi(f)^{1/2} \geq \psi(f^{1/2}) \geq \psi(a) - \delta > 0$.

There exists a separable C^*-subalgebra G of A such that d is in the closure of $G \odot B$, because d is the limit of sequence in $A \odot B$. By Lemma 2.14, there exists $b \in A_+$ such that $\|b\| = 1 = \varphi(b)$ and $\|b^n c b^n - \varphi(c)b^{2n}\|$ tends to zero for every $c \in G$. The maps

$$T_n\colon\ y \in A \otimes B \mapsto (b^n \otimes 1)y(b^n \otimes 1) - (b^{2n} \otimes (\varphi \otimes \mathrm{id})(y))$$

converge on $G \otimes B$ pointwise to zero, because T_n is a difference of completely positive contractions on C and tends on $G \odot B$ pointwise to zero.

Thus, there exists n with $\|T_n(d)\| < \delta^2$, i.e., $(b^{2n} \otimes a) - \delta^2 \leq (b^n \otimes 1)d(b^n \otimes 1)$ in the unitization of C. With $g := b^{2n}$ and $t := d^{1/2}(b^n \otimes 1)(g \otimes a) - \delta^2)_+^{1/2}$, we get $((g \otimes a) - \delta^2)_+^2 \leq t^*t$ and $tCt^* \subset d^{1/2}Ad^{1/2} \subset D$.

Now let $e := (g - \delta)_+{}^2 \in A_+$. Since $\varphi(g) = \|g\| = 1$, $g \geq 0$ and $0 < \delta \leq 1/2$, we get $\varphi(e) \geq (1 - \delta)^2 > 0$.

On the other hand, $e \otimes f \leq ((g \otimes a) - \delta^2)_+)^2$ by functional calculus.

If $t = (tt^*)^{1/2}v$ is the polar decomposition of t in the second conjugate of C, then $vxv^* \in \overline{tCt^*} \subset D$ and $vx^{1/2} \in C$ for every $x \in C$ with $0 \leq x \leq t^*t$, because $x^{1/2}$ is in the norm closure of t^*Ct and $vt^* = (tt^*)^{1/2}$.

Since $e \otimes f \leq t^*t$ we get that $z = v((g - \delta)_+ \otimes (a - \delta)_+)$ is in $A \otimes B$, and e, f, z satisfy $z^*z = e \otimes f$, $zz^* \in D$ and $\varphi(e)\psi(f) > 0$. $\square$

Proposition 2.16. *Given two C^*-algebras A and B, the following conditions (i)-(iv) are equivalent:*

(i) *For every primitive ideal $I \triangleleft A \otimes B$ and every $d \in (A \otimes B)_+ \setminus I$ there are pure states φ on A and ψ on B such that $(\varphi \otimes \psi)(I) = 0$ and $(\varphi \otimes \psi)(d) > 0$.*

(ii) *Every closed ideal J of $A \otimes B$ is the closure of the sum of all elementary ideals $J_1 \otimes J_2 \subset J$, where $J_1 \subset A$ and $J_2 \subset B$ are closed ideals.*

(iii) *The map*

$$\lambda \colon (J_1, J_2) \mapsto (J_1 \otimes B) + (A \otimes J_2)$$

defines a homeomorphism from the Tychonoff product $\mathrm{prime}(A) \times \mathrm{prime}(B)$ *of* $\mathrm{prime}(A)$ *and* $\mathrm{prime}(B)$ *onto* $\mathrm{prime}(A \otimes B)$.

(iv) *For every closed ideals $I \triangleleft A$ and $J \triangleleft B$ the sequences*

$$I \otimes (B/J) \to A \otimes (B/J) \to (A/I) \otimes (B/J)$$

and

$$(A/I) \otimes J \to (A/I) \otimes B \to (A/I) \otimes (B/J)$$

are exact.

Clearly A must be exact if A and B satisfy (iv) for every C^*-algebra B.

Proof. The implication (ii)⇒(iv) follows from part (iii) of Lemma 2.12.

(iv)⇒(i): The primitive ideal I is prime. By Lemma 2.13, $I_A \otimes B + A \otimes I_B \subset I$ and I is contained in the kernel of $A \otimes B \to (A/I_A) \otimes (B/I_B)$. By the 3×3-lemma it follows from (iv) that $A \otimes B \to (A/I_A) \otimes (B/I_B)$ has kernel $I_A \otimes B + A \otimes I_B$. Thus, $I = A \otimes I_B + I_A \otimes B$ is the kernel of $A \otimes B \to (A/I_A) \otimes (B/I_B)$. Since $d \notin I$, the image $d + I$ is a non-zero positive element of $(A/I_A) \otimes (B/I_B)$. The irreducible representations $\rho_1 \otimes \rho_2$ for irreducible representations ρ_1 of A/I_A and ρ_2 of B/I_B are separating for $(A/I_A) \otimes (B/I_B)$. Thus, there are pure states φ on A and ψ on B such that $\varphi(I_A) = 0$, $\psi(I_B) = 0$ and $(\varphi \otimes \psi)(d) > 0$.

(i)⇒(ii): Let J_0 be the closure of the sum of all elementary ideals which are contained in J.

Suppose that there exists $d \in J_+$ such that d is not in J_0. Then there is an irreducible representation ρ of $A \otimes B$ with primitive kernel I such that $J_0 \subset I$ but $d \notin I$. By (i) there are pure states φ on A and ψ on B with $\varphi \otimes \psi(I) = 0$ and $\varphi \otimes \psi(d) > 0$. There exist $e \in A_+$ and $f \in B_+$ such that $\varphi(e)\psi(f) > 0$ and $e \otimes f$ is in J by Lemma 2.15. Thus, $e \otimes f \in I$, which contradicts $(\varphi \otimes \psi)(I) = 0$.

(ii)⇒(iii): The map λ is well-defined: the kernel I of $A \otimes B \to (A/J_1) \otimes (B/J_2)$ is prime for prime $J_1 \triangleleft A$ and prime $J_2 \triangleleft B$ by part (v) of Lemma 2.13. Thus, (ii) implies $I = A \otimes J_2 + J_1 \otimes B$ by part (iii) of Lemma 2.12,i.e., λ is a well-defined map from prime$(A) \times$ prime(B) into prime$(A \otimes B)$.

If I is a given prime ideal of $A \otimes B$ then $I_A \triangleleft A$ and $I_B \triangleleft B$ are primitive ideals and the the kernel K of $A \otimes B \to (A/I_A) \otimes (B/I_B)$ is prime and contains I by Lemma 2.13. Thus, (ii) implies $I = K = I_A \otimes B + A \otimes I_B$ by part (iii) of Lemma 2.12, i.e., λ is onto. Part (v) of Lemma 2.13 shows that the inverse of λ is given by $I \mapsto (I_A, I_B)$.

Every open subset of the Tychonoff product prime$(A) \times$ prime(B) is the union of Cartesian products $Z_J \times Z_K$ of open subsets Z_J of prime(A) and Z_K of prime(B) corresponding to closed ideals $J \triangleleft A$ and $K \triangleleft B$. λ maps $Z_J \times Z_K$ onto the open subset of prime$(A \otimes B)$ which corresponds to $J \otimes K$. Thus, by (ii), λ maps the open subsets of the Tychonoff product prime$(A) \times$ prime(B) onto the open subsets of prime$(A \otimes B)$.

(iii)⇒(ii): Follows from the correspondence of open sets of prime$(A \otimes B)$ and closed ideals of $A \otimes B$. □

Proposition 2.17. *Given two C^*-algebras A and B, each of the following properties (1)-(5) imply the equivalent properties (i)-(iv) in Proposition 2.16.*

1. *For $I \triangleleft A$ and $J \triangleleft B$ the sequences $\mathcal{L}(\mathcal{H}) \otimes I \to \mathcal{L}(\mathcal{H}) \otimes A \to \mathcal{L}(\mathcal{H}) \otimes (A/I)$ and $\mathcal{L}(\mathcal{H}) \otimes J \to \mathcal{L}(\mathcal{H}) \otimes B \to \mathcal{L}(\mathcal{H}) \otimes (B/J)$ are exact.*
2. *A or B is exact*
3. *A and B are locally reflexive*
4. *B is simple and for every ideal $I \triangleleft A$ the sequence $I \otimes B \to A \otimes B \to (A/I) \otimes B$ is exact.*
5. *A is locally reflexive and B is simple*
6. *A and B are simple.*

Proof. (1): The exactness of the sequence $\mathcal{L}(\mathcal{H}) \otimes I \to \mathcal{L}(\mathcal{H}) \otimes A \to \mathcal{L}(\mathcal{H}) \otimes (A/I)$ implies that for every C^*-algebra C the sequence $C \otimes I \to C \otimes A \to C \otimes (A/I)$ is exact, cf. [62, prop. 2.6] or [34, lemma 3.9].

(2): If A is exact, then every quotient A/I of A is exact (cf. [35, prop. 7.1(ii)] or [62, cor.9.3]), i.e., the sequence $(A/I) \otimes J \to (A/I) \otimes B \to (A/I) \otimes (B/J)$ is exact. Every exact C^*-algebra is locally reflexive (cf. [35, rem. above thm. 7.2] or [62, prop.5.1]), which implies the exactness of $\mathcal{L}(\mathcal{H}) \otimes I \to \mathcal{L}(\mathcal{H}) \otimes A \to \mathcal{L}(\mathcal{H}) \otimes (A/I)$ for every $I \triangleleft A$ by [26]. Thus, $I \otimes (B/J) \to A \otimes (B/J) \to (A/I) \otimes (B/J)$ is exact for every $I \triangleleft A$.

(3): If A is locally reflexive, then $\mathcal{L}(\mathcal{H}) \otimes I \to \mathcal{L}(\mathcal{H}) \otimes A \to \mathcal{L}(\mathcal{H}) \otimes (A/I)$ is exact for $I \triangleleft A$, see [26].

(5) implies (4) in the same way. (6) implies (4), and (4) implies (iv) of Proposition 2.16. □

Lemma 2.18. *Suppose that the natural map from* prime$(A) \otimes$prime(B) *into* prime$(A \otimes B)$ *is an isomorphism, D is a hereditary C^*-subalgebra of $A \otimes B$ and I is a primitive* *ideal of $A \otimes B$ which does not contain D.*

Then there are non-zero $g \in A_+$, $h \in B_+$, $t \in A \otimes B$ and pure states φ on A and ψ on B such that

(i) $(\varphi \otimes \psi)(I) = 0$,
(ii) $tt^* \in D$, $t^*t = g \otimes h$,
(iii) $\varphi(g) = \|g\| = 1$ *and* $\psi(h) = \|h\| = 1$.

Proof. There exists $d \in D_+ \setminus I$. By (i) of Proposition 2.16 there are pure states φ_0 on A and ψ_0 on B such that $(\varphi_0 \otimes \psi_0)(I) = 0$ and $(\varphi_0 \otimes \psi_0)(d) > 0$. By Lemma 2.15 there are $z \in A \otimes B$, $e \in A_+$, $f \in B_+$ with $zz^* \in \overline{d(A \otimes B)d}$, $z^*z = e \otimes f$, $\varphi_0(e) > 0$ and $\psi_0(f) > 0$.

Let $\varphi(a) := \varphi_0(e^{1/2}ae^{1/2})/\varphi_0(e)$ and $\psi(b) := \psi_0(f^{1/2}bf^{1/2})/\psi_0(f)$. Then φ and ψ are pure states on A respectively B. The restrictions to $\overline{eAe}$ respectively $\overline{fBf}$ have norm one. Thus, the restrictions are pure states on $\overline{eAe}$ respectively $\overline{fBf}$. By Lemma 2.14 there are $g \in \overline{eAe}$ and $h \in \overline{fBf}$ which satisfy (iii). Now let $z = w(e^{1/2} \otimes f^{1/2})$ be the polar decomposition of z in the second conjugate of $A \otimes B$ and define t by

$$t := w(g^{1/2} \otimes h^{1/2}) = \lim_{n\to\infty} z(((e + 1/n)^{-1/2}g^{1/2}) \otimes (f + 1/n)^{-1/2}h^{1/2})) \in A \otimes B\,.$$

□

Lemma 2.19. *Let $b \in A_+$ be a positive element such that, for every non-negative function $f \in C_0((0, \|b\|])$, there is no non-zero tracial state on $\overline{f(b)Af(b)}$.*

Then for every $\delta \in (0, 1)$ and $n \in \mathbb{N}$ there are elements $d_1, \ldots, d_n \in A^{**}$ such that $\|b - \sum d_j^* b d_j\| \leq \delta$, $\sum \|d_j^* d_j\| \leq \delta^{-1}\|b\|$ and $\| \sum d_j d_j^*\| \leq (n\delta)^{-1}\|b\|$.

Proof. We may suppose $b \neq 0$. For $\delta \in [0, \|b\|)$ let p denote the support projection of $(b - \delta)_+$ in A^{**}. Then $\|bp - b\| \leq \delta$. Since p is the unit element of the second conjugate of the closure of $(b-\delta)_+A(b-\delta)_+$ and since $\overline{(b - \delta)_+A(b - \delta)_+)}$ has no non-zero tracial state, p is a properly infinite projection in A^{**}, which commutes with b, and $(1 - p) + b^{1/2}p$ has an inverse $c \in A^{**}$ of norm $\leq \delta^{-1/2}$. Thus, $\psi(1) = p$ for some *-morphism $\psi\colon \mathcal{O}_\infty \to A^{**}$. If $s_1, s_2, \ldots$ are the canonical generators of $\mathcal{O}_\infty$ and $n \in \mathbb{N}$ let $d_j := n^{-1/2}c\psi(s_j)b^{1/2}$ for $j = 1, \ldots, n$. Then $\sum d_j^* b d_j = bp$ and $\sum d_j d_j^* \leq n^{-1}\|b\|c^2$. In particular $\|d_j^* d_j\| \leq \| \sum d_j d_j^*\| \leq (n\delta)^{-1}\|b\|$. □

Remark 2.20. It is likely that one can prove the following stronger result: suppose that b is a positive element in a von Neumann algebra M such that for every projection p in the center of M and every $c \in C^*(b)_+$ the support projection of cp is zero or infinite in M. Then for every $\varepsilon > 0$ there are partial isometries $s, t \in M$ such that $s^*s = t^*t = ss^* + tt^*$, $\|bs - sb\| < \varepsilon$ $\|bt - tb\| < \varepsilon$, $\|s^*bt\| < \varepsilon$ and s^*s is the support projection of b in M.

The following lemma is a generalization of [27, thm. 2.4]. Its assumption means that every lower semi-continuous additive trace $\tau\colon A_+ \to [0, +\infty]$ takes only the values 0 and $+\infty$.

Lemma 2.21. *Suppose that for every hereditary C^*-subalgebra D of A every tracial positive linear functional on D is zero. Then for every $a \in A_+$ and $\mu > 0$ the element $(a - \mu)_+ \otimes 1$ is properly infinite or zero in $A \otimes C^*_{\mathrm{red}}(F_2)$.*

Proof. Fix $\mu \in (0, \|a\|)$ and let $b := (a-\mu)_+$. We consider the set X of finite sequences $d_1, \ldots, d_n \in A$, $n \in \mathbb{N}$. Define $\kappa(d_1, \ldots) \in A^2 := A \oplus A$ by $(b - \sum d_j^* b d_j, \sum d_j d_j^*)$. The set X can be considered in different ways as a dense subset of the standard Hilbert-A-module $\mathcal{H}_A$. This allows to check that

(i) the image of $\kappa \colon X \to A^2$ is convex,
(ii) if one defines $\kappa \colon Y \to (A^{**})^2 \cong (A^2)^{**}$ similarly for the set of finite sequences Y in $\mathcal{H}_{A^{**}}$, then $\kappa(X)$ is weakly dense in $\kappa(Y)$, and
(iii) The norm-closure of $\kappa(Y)$ contains zero by Lemma 2.19.

Thus, we can use a Hahn-Banach separation argument, to deduce from (i)–(iii) that, for $\varepsilon \in (0,1)$, there are $n \in \mathbb{N}$ and $d_1, \ldots, d_n \in A$ such that $\|b - \sum d_j^* b d_j\| < \varepsilon$ and $\|\sum d_j d_j^*\| < \varepsilon^2$.

Let $s_1, s_2, \ldots$ be the canonical generators of $\mathcal{O}_\infty$. Consider the elements $f_1 := d_1 \otimes s_1 + \ldots + d_n \otimes s_n$, $f_2 := d_1^* \otimes s_1 + \ldots + d_n^* \otimes s_n$, $g_1 := d_1 \otimes s_{n+1} + \ldots + d_n \otimes s_{2n}$ and $g_2 := d_1^* \otimes s_{n+1} + \ldots + d_n^* \otimes s_{2n}$ of $A \otimes \mathcal{O}_\infty$. Then $\|f_2\|^2 = \|\sum d_j d_j^*\| < \varepsilon^2$, $\|g_2\| < \varepsilon^2$, $\|b \otimes 1 - f_1^*(b \otimes 1) f_1\| = \|b - \sum d_j^* b d_j\| < \varepsilon$, $\|b \otimes 1 - g_1^*(b \otimes 1) g_1\| < \varepsilon$ and $f_1^*(b \otimes 1) g_1 = 0$. This implies that $\|(b^{1/2} \otimes 1) f_1\|^2 < 1 + \|b\|$ and $\|(b^{1/2} \otimes 1) g_1\|^2 < 1 + \|b\|$.

The elements $b \otimes 1$, $f := f_2^* + f_1 = \sum d_j \otimes (s_j^* + s_j)$ and $g := g_2^* + g_1$ are in the C^*-subalgebra $A \otimes C^*(1, x_1, x_2, \ldots)$ of $A \otimes \mathcal{O}_\infty$, where $x_n := (s_n^* + s_n)/2$ for $n \in \mathbb{N}$.

As pointed out in [27], $C^*(1, x_1, x_2, \ldots)$ is naturally isomorphic to the infinite reduced free product $\mathcal{V}_\infty$ of $C([-1,1])$ (with respect to the semicircular state on it) and there are unital embeddings $\mathcal{V}_\infty \subset C_r^*(F_\infty) \subset C_r^*(F_2)$. Thus, $b \otimes 1$, f and g can be considered as elements of $A \otimes C_r^*(F_2)$.

The above estimates show $\|b \otimes 1 - f^*(b \otimes 1) f\| < 3(\|b\|+1)\varepsilon$, $\|b \otimes 1 - g^*(b \otimes 1) g\| < 3(\|b\|+1)\varepsilon$ and $\|f^*(b \otimes 1) g\| < (3 + 2\|b\|)\varepsilon$. Since $\varepsilon \in (0,1)$ was arbitrary, the element $b \otimes 1$ is properly infinite in $A \otimes C_r^*(F_2)$. □

2.9. **Quasi-traces.**

In order to study the different possible generalizations of pure infiniteness to the non-simple case, let us recall some definitions of Blackadar, Cuntz, Haagerup and Handelman ([16], [6], [27]), which we modify for our needs. Later we prefer to work with lower semi-continuous *quasi-traces* (in the sense of Definition 2.22). Therefore we outline some results concerning characterizations of 2-quasi-traces and "traceless" algebras. Some results (e.g. of Haagerup) and open problems are mentioned.

Definition 2.22. A *local quasi-trace* on a C^*-algebra A is a function $\tau \colon A_+ \to [0, \infty] = \mathbb{R}_+ \cup \{\infty\}$ which satisfies $\tau(d^*d) = \tau(dd^*)$ for all $d \in A$ and $\tau(a+b) = \tau(a) + \tau(b)$ if there is a self-adjoint element $f \in A$ such that the two positive elements $a, b \in A_+$ are in the C^*-subalgebra $C^*(f)$ of A.

The local quasi-trace τ is said to be:
– a *quasi-trace* if $\tau(a+b) = \tau(a) + \tau(b)$ for all commuting positive elements $a, b \in A_+$;
– a 2-*quasi-trace* if it extends to a quasi-trace τ_2 on $M_2(A)$ with $\tau_2(a \otimes e_{1,1}) = \tau(a)$ for all $a \in A_+$;
– *trivial* if it takes only the values 0 and ∞. (We call the C^*-algebra A *traceless* if every lower semi-continuous 2-quasi-trace on A is trivial.);

– *semi-finite* if the set $\mathrm{Dom}_{1/2}(\tau) := \{a \in A;\ \tau(a^*a) < \infty\}$ is dense in A;
– *bounded* if $\mathrm{Dom}_{1/2}(\tau) = A$, i.e., $\tau(A_+) \subset [0, \infty)$;
– *faithful* if $\tau(a) > 0$ for non-zero $a \in A_+$, i.e., the set $I_\tau := \{a \in A;\ \tau(a^*a) = 0\}$ is 0;
– *locally lower semi-continuous* if $\tau(a) = \sup_{t>0} \tau((a-t)_+)$ for $a \in A_+$ (Then τ is order preserving and lower semi-continuous on A_+ in the norm-topology on A_+ by Remarks 2.27(iii) and (iv).)

Every bounded local quasi-trace (respectively quasi-trace) τ on A_+ extends uniquely to a uniformly continuous map $\tau_e \colon A \to \mathbb{C}$ such that $\tau_e(a) = \tau(a)$ for $a \in A_+$, $\tau_e(b^*) = \overline{\tau_e(b)}$ for $b \in A$ and τ_e is linear on every C^*-subalgebra $C^*(h)$ of A for selfadjoint $h \in A$ (respectively τ_e is linear on every commutative C^*-subalgebra of A). If τ is a bounded quasi-trace τ_e fulfills the original definition [6, def. II.1.1] of quasi-traces (respectively of 2-quasi-traces if τ is a 2-quasi-trace). But it follows from [1] that there is a bounded local quasi-trace τ_A on $C([0,1] \times [0,1])_+$ which is not additive and does not have additively closed I_τ. By Proposition 2.25, this also implies the existence of a bounded quasi-trace which is not a 2-quasi-trace.

We introduce a function $t \mapsto Q(\tau, t) \in [0, \infty]$ for local quasi-traces τ and $t \in (0, 1]$:

$$Q(\tau, t) := \sup\{\tau(a + b);\ a, b \in A_+, \|a\|, \|b\| \le 1, \tau(a) + \tau(b) \le t\} \tag{2.6}$$

Then $Q(\tau, t)$ is increasing in t and $\tau(a+b) \le (\inf_t Q(\tau, t)/t) \max\{\|a\|, \|b\|, \tau(a)+\tau(b)\}$.

Definition 2.23. A (not necessarily bounded) *local rank function* on a C^*-algebra A is a function $D \colon A_+ \to [0, \infty]$ which satisfies the following conditions (1)-(5):

(1) $D(a) = D(a^*a) = D(a^*) = D(ta)$ for $a \in A$ and $t > 0$,
(2) $D(e) \le D(f) = D(f + e)$ for $0 \le f$, $0 \le e = ef$,
(3) $D(b) = D(b_+) + D(b_-)$ for $b = b^*$,
(4) D is locally lower semi-continuous, i.e., $D(a) = \sup_{\delta>0} D((a - \delta)_+)$ for $a \in A_+$, and
(5) D is *locally sub-additive*, i.e., $D(c + d) \le D(c) + D(d)$ if there exists some selfadjoint $b \in A$ such that c, d are both in $C^*(b)_+ \subset A$.

A local rank function D is said to be:
– a (lower semi-continuous, unbounded) *rank function* if D is *weakly sub-additive*, i.e., $D(c + d) \le D(c) + D(d)$ for all commuting $c, d \in A_+$;
– *sub-additive* if $D(c + d) \le D(c) + D(d)$ for all $c, d \in A_+$;
– *bounded* (respectively *trivial*, *semi-finite*, *faithful*) if $D(A) \subset [0, \infty)$ (respectively $D(A) \subset \{0, \infty\}$, $D((a - \varepsilon)_+) < \infty$ for $a \in A_+$ and $\varepsilon > 0$, $D(a) > 0$ for $a \neq 0$).

A (lower semi-continuous, semi-finite) *dimension function* on A means a semi-finite local rank function on $A \otimes \mathcal{K}$.

Our rank functions are (unbounded and lower semi-continuous) generalizations of the bounded weakly sub-additive rank functions defined in [6], cf. Remark 2.27(ii). A dimension function is automatically a sub-additive rank function on $A \otimes \mathcal{K}$, cf. Remark 2.27(viii), and is determined by its restrictions to $\bigcup_n M_n(A_{\min})$, where $A_{\min}$ denotes the Pedersen ideal, i.e., the minimal dense ideal of A. There D takes values in $[0, \infty)$. This restriction satisfies the axioms for a dimension function in [16] except the existence of a full hereditary C^*-subalgebra B of A where $D|B$ is finite (B exists automatically in the case of simple A).

We introduce here a later often needed function $g_\delta \in C_0((0,\infty])$ which is given by

$$g_\delta(t) = \begin{cases} 0 & \text{if } 0 \le t \le \delta, \\ (t-\delta)/\delta & \text{if } \delta \le t \le 2\delta \\ 1 & \text{if } 2\delta \le t \end{cases} \tag{2.7}$$

For every local quasi-trace τ the formula

$$D_\tau(a) := \sup_{\delta>0} \tau(g_\delta((a^*a)^{1/2})) = \sup_{t>0} \lim_{n\to\infty} \tau((a^*a) - t)_+^{1/n}) \,. \tag{2.8}$$

defines a local rank function on A, cf. Remark 2.27(iv). Conversely local rank functions D on A_+ define locally lower semi-continuous local quasi-traces τ_D by

$$\tau_D(a) := \int_{0+}^{\infty} D((a-t)_+)dt \,, \tag{2.9}$$

where $\tau_D(a) := \infty$ if $D((a-t)_+) = \infty$ for some $t > 0$. Moreover τ_D is additive on the positive elements in commutative C^*-subalgebras $C \subset A$ if D is sub-additive on C, cf. Remark 2.27(iii). Thus, τ_D is a quasi-trace if D is weakly sub-additive. A look to the related outer Caratheodory-Radon measures (on the open subsets of $\mathrm{Spec}(a) \setminus \{0\}$) shows that for locally l.s.c. local quasi-traces τ and for arbitrary local rank functions D the following holds:

$$\tau = \tau_{D_\tau} \qquad \text{and} \qquad D = D_{\tau_D} \,. \tag{2.10}$$

In the following proposition a *local AW*-algebra* means a C^*-algebra B of real rank zero such that pBp is an AW*-algebra for every projection $p \in B$.

Proposition 2.24. *Let $\tau\colon A_+ \to [0,\infty]$ be a locally lower semi-continuous local quasi-trace. Then the following are equivalent:*

(i) *τ is a lower semi-continuous 2-quasi-trace.*
(ii) *$\tau(a+b)^{1/2} \le \tau(a)^{1/2} + \tau(b)^{1/2}$ for every $a, b \in A_+$.*
(iii) *$\tau(a+b) \le 2(\tau(a) + \tau(b))$ for every $a, b \in A_+$.*
(iv) *$\inf_{t>0} Q(\tau, t) = 0$.*
(v) *There are a closed ideal I of A, a *-homomorphism φ from I into a local AW*-algebra B and a faithful semi-finite lower semi-continuous quasi-trace τ_1 on B_+ such that $\tau(a) = \tau_1(\varphi(a))$ for $a \in I_+$ and $\tau(a) = \infty$ for $a \in A_+ \setminus I$.*
(vi) *There is a closed ideal I of A and *-homomorphism ψ from I into a C^*-algebra C of real rank zero and a locally lower semi-continuous local quasi-trace τ_1 on C_+ such that $\tau(a) = \tau_1(\psi(a))$ for $a \in I_+$ and $\tau(a) = \infty$ for $a \in A_+ \setminus I$.*
(vii) *The local rank function D_τ of τ is sub-additive.*
(viii) *There exists $\kappa > 0$ such that $\kappa D_\tau(a+b) \le D_\tau(a) + D_\tau(b)$ for $a, b \in A$.*
(ix) *The closure J of $Dom_{1/2}(\tau)$ is an ideal and there is a unique dimension function $D\colon \bigcup M_n(J_{\min}) \to [0,\infty)$, where $J_{\min}$ is the Pedersen ideal of J, such that $\tau(a) = \int_{0+}^{\infty} D((a-t)_+ \otimes e_{1,1})dt$ for $a \in J_+$ and $\tau(a) = \infty$ if $a \in A_+ \setminus J$.*

Clearly Proposition 2.24 implies that l.s.c. dimension functions correspond 1-1 to l.s.c. 2-quasi-traces, that all bounded sub-additive rank functions or 2-quasi-traces on

A come from homomorphisms from A into finite AW*-algebras, that every locally l.s.c. local quasi-trace on a C^*-algebra of real rank zero is a l.s.c. 2-quasi-trace, and *that our definition of "A is traceless" in 2.22 is equivalent to* [45, def. 4.2], cf. Remark 2.27(viii).

Proposition 2.25. *Let $C([0,1]) * C([0,1])$ denote the unital full free C^*-algebra product and let π be the natural *-epimorphism from $C([0,1]) * C([0,1])$ onto $C([0,1] \times [0,1])$.*

Then for every bounded local quasi-trace τ on $C([0,1] \times [0,1])$ the lift $\tau\pi$ is a bounded quasi-trace on $C([0,1]) * C([0,1])$

In particular, $\tau_A\pi$ is a bounded quasi-trace which is not a 2-quasi-trace, and there is a trivial l.s.c. quasi-trace which is not a trace.

(Here τ_A on $C([0,1] \times [0,1])$ is a non-linear quasi-state as defined in [1].) A proof of Propositions 2.24 and 2.25 will be given in [41]. But we outline in Remarks 2.28 the non-trivial parts of the proof of 2.24.

Lemma 2.26. *If $c, d \in A_+$ and $\eta \geq 0$ satisfy $c - \eta \leq d$ then for every $\delta > 0$ and every function $f \in C_0((0, \|c\|])_+$ there is $x \in A$ such that $x^*x = f((c - \eta - 2\delta)_+)$, $xx^*g_\delta(d) = xx^*$.*

Proof. Let $e := (c - \eta - 2\delta)_+$, $h := (d - 2\delta)_+ e$ and $h = v(h^*h)^{1/2}$ the polar-decomposition of h in the second conjugate A^{**} of A. The element $x := vf(e)^{1/2}$ is as desired, because $g_\delta(d)h = h$ and so $g_\delta(d)x = x$, $|h|v^*v|h| = |h|^2$ and $f(e)$ is in $\overline{|h|A}$. □

Remarks 2.27. (i) The property (1) of Definition 2.23 reduces all considerations on local rank functions to positive elements. It says also $D(za) = D(a)$ for complex $z \neq 0$. (3) implies $D(0) = 0$.

Lemma 2.26 and (1)-(5) yield further properties of local rank functions:

(6) $D(a) \leq \sup\{D(b_n);\, n \in \mathbb{N}\}$ if $a, b_1, b_2, \ldots \in A_+$ and for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$, $\delta > 0$ and $x \in A$ such that $xx^* = (a - \varepsilon)_+$, $g_\delta(b_n)x = x$.
(7) $D(f(b)) \leq D(b)$ for $b \in A_+$, $f \in C_0((0, \|b\|])$.
(8) $D(a) \leq D(b)$ for $0 \leq a \leq b$.
(9) D is lower semi-continuous: $D(a) \leq \sup_n\{D(a_n)\}$ if a_n converges to a.
(10) $D(a) \leq D(b)$ if a is in the closure of the set $\{xby,\, x, y \in A\}$.
(11) $D(a + b) = D(a) + D(b)$ for $a, b \in A$ with $b^*a = 0 = ab^*$.
(12) $D(ab) \leq \min(D(a), D(b))$ for $a, b \in A$.

D is a sub-additive rank function if there is a local rank function D_2 on $M_2(A)$ with $D(a) = D_2(a \otimes e_{1,1})$ for $a \in A$. (by (10) and [6, prop 1.1.7]).

(ii) If $D(A) \subset [0, \infty)$, i.e., if D is "bounded", then $\|D\| := \sup\{D(a);\, a \in A\} < \infty$, as follows from (1) and (8) by an obvious indirect argument. Thus, *bounded and weakly sub-additive local rank functions are, up to normalization, just the rank functions in the sense of* [6, def. I.1.2].

(iii) A local rank function $D\colon A_+ \to [0, \infty]$ defines a local quasi-trace $\tau_D\colon A_+ \to [0, \infty]$ by the formula (2.9). τ_D is order preserving and lower semi-continuous with respect to the norm-topology on A_+ and is additive on the positive part of every commutative

C^*-subalgebra $C \subset A$ on which D is sub-additive. In particular τ_D is a quasi-trace if D is weakly sub-additive.

(iv) If τ is a local quasi-trace on A then $\tau_*(a) := \sup_{\delta>0} \tau((a-\delta)_+)$ is a locally l.s.c. local quasi-trace on A_+ and the formula (2.8) defines a map D_τ from A into $[0, +\infty]$ which satisfies (1)-(5) of the Definition 2.23. It holds $D_\tau = D_{\tau_*}$, $\tau_{D_\tau} = \tau_*$. Thus, τ_* is order preserving and lower semi-continuous on A_+ by Remark (iii). It is a quasi-trace (respectively 2-quasi-trace) if τ is a quasi-trace (respectively 2-quasi-trace). In general $\mathrm{Dom}_{1/2}(\tau) \subset \mathrm{Dom}_{1/2}(\tau_*)$, the norm closure of $\mathrm{Dom}_{1/2}(\tau)$ contains $\mathrm{Dom}_{1/2}(\tau_*)$ and $\tau((a-\delta)_+) < \infty$ for every positive element a in the norm closure of $\mathrm{Dom}_{1/2}(\tau)$ and every $\delta > 0$.

(v) Bounded local quasi-traces τ are order preserving and lower semi-continuous, because they are automatically locally lower semi-continuous. This allows to see that $\|\tau\| := \sup\{\tau(a);\, a \in A_+, \|a\| \leq 1\} < \infty$, $\tau(a) \leq \|\tau\| \cdot \|a\|$ and $|\tau(a) - \tau(b)| \leq \|\tau\| \cdot \|a - b\|$ for $a, b \in A_+$. Then τ extends to a uniformly continuous function on A by $\tau_e(a + ib) := \tau(a_+) - \tau(a_-) + i\tau(b_+) - i\tau(b_-)$ for selfadjoint $a, b \in A$. If A is unital and $\tau(1) = 1$ then τ_e is a (central) quasi-state in the sense of [1].

(vi) If τ is a 2-quasi-trace, then τ satisfies

$$\tau(a+b)^{1/2} \leq \tau(a)^{1/2} + \tau(b)^{1/2} \quad \text{for} \quad a, b \in A_+\,,$$

(the proof of [27, lem. 3.5(1)] works also in the unbounded case). It follows that τ is 2-*additive*, i.e.,

$$\tau(a+b) \leq 2(\tau(a) + \tau(b)) \text{ for } a, b \in A_+\,,$$

hence the set $I_\tau := \{a \in A;\, \tau(a^*a) = 0\}$ is a closed two-sided ideal in A, and $\mathrm{Dom}_{1/2}(\tau)$ is an algebraic *-ideal of A and the Pedersen ideal $J_{\min}$ of the closure J of $\mathrm{Dom}_{1/2}(\tau)$ is contained in the set $\{a \in A : D_\tau(a) < \infty\}$. In general $\mathrm{Dom}_{1/2}(\tau)$ is not a subset of $\{a \in A : D_\tau(a) < \infty\}$.

(vii) The following elementary reductions to the unital case is inspired by [37] and are easily verified. One could also use results of [6] and extensions of bounded sub-additive rank functions.

If A is not unital, τ is a bounded quasi-trace on A_+ and c is a positive contraction in the center of A which is strictly positive for A, then $\widetilde{\tau}(b) := \sup_n \tau(c^{1/n}b)$ is an extension of τ to a bounded quasi-trace on $\mathcal{M}(A)_+$. The extension $\widetilde{\tau}$ is a 2-quasi-trace, (respectively an additive trace) if τ is such. The extension τ_2 to $M_2(A)_+$ is unique if the extension $\widetilde{\tau}_2$ to $M_2(\mathcal{M}(A))$ is unique. (Note that $c \otimes 1_2$ defines $\widetilde{(\tau_2)}$.)

This together with Remark (iv) implies the following.
Let τ be an unbounded lower semi-continuous quasi-trace on A_+, then τ is a 2-quasi-trace (respectively is an additive trace) if and only if the closure of $\mathrm{Dom}_{1/2}(\tau)$ is an ideal and, for every $b, c, d \in A_+$ with $bc = b$, $cd = c$, $\|d\| \leq 1$ and $\tau(d) < \infty$, the extension to $(\mathbb{C}1 + C^*(bAb, c))_+ \subset \mathcal{M}(C^*(bAb, c))$ of the restriction of τ to $C^*(bAb, c) \subset A$ (as defined above) is a 2-quasi-trace (respectively is an additive trace). Moreover a l.s.c. extension τ_n of τ to $M_n(A)_+$ with $\tau(a) = \tau_n(a \otimes e_{1,1})$ is unique if all the extensions of the local restrictions have unique extensions to $M_n(\mathbb{C}1 + C^*(bAb, c))_+$.

(viii) If one uses the above reduction to the unital case and [6] or the ultrapowers of τ as in the below given Remarks 2.28(ix) and compares them with the ultrapowers of its possible extension to $M_n(A)_+$, then one gets the following results on extensions of l.s.c. 2-quasi-traces and its uniqueness.

For every l.s.c. 2-quasi-trace τ on A_+ and every $n \in \mathbb{N}$ there is a *unique* l.s.c. 2-quasi-trace τ_n on $M_n(A)_+$ with $\tau(a) = \tau_n(a \otimes e_{1,1})$ for $a \in A_+$.

Let J denote the closure of $\mathrm{Dom}_{1/2}(\tau)$ and $J_{\min}$ the minimal dense ideal (Pedersen ideal) of J. By Lemma 2.26, for every element b in the Pedersen ideal $(J \otimes \mathcal{K})_{\min}$ of $J \otimes \mathcal{K}$ there is $n \in \mathbb{N}$, a positive contraction $c \in J_+$, $\delta > 0$ and a contraction $d \in J \otimes \mathcal{K}$ with $dd^*(g_\delta(c) \otimes 1) = dd^*$ and $d^*db = b = bd^*d$. Since $\tau((a-\delta)_+) < \infty$ and $D_\tau((a-\delta)_+) < \infty$ for $a \in J_+$ one obtains that *there are a unique semi-finite l.s.c. quasi-traces $\overline{\tau}$ on $(J \otimes \mathcal{K})_+$ and a unique semi-finite l.s.c. dimension function D on $J \otimes \mathcal{K}$ such that $\tau(a) := \overline{\tau}(a \otimes e_{1,1})$ and $D_\tau(a) = D(a \otimes e_{1,1})$ for $a \in J_+$*. Moreover, $\overline{\tau}(c) < \infty$ and $D(d) < \infty$ for $c, d \in (J \otimes \mathcal{K})_{\min}$, $c \geq 0$, and D is determined by its restriction $D_|$ to $\bigcup_n M_n(J_{\min})$, takes there finite values. $D_|$ is l.s.c. and satisfies the requirements of a dimension function in [16] (except the existence of a *full* hereditary C^*-subalgebra of J where $D_|$ is bounded).

Thus, τ (respectively D_τ) is determined by J and $\overline{\tau}\colon (J \otimes \mathcal{K})_+ \to [0,\infty]$ (respectively $D\colon J \otimes \mathcal{K} \to [0,\infty]$) because $\tau(b) = D_\tau(b) = \infty$ for $b \in A_+ \setminus J$.

Since the l.s.c. sub-additive rank functions D with finite values on the Pedersen ideal $J_{\min}$ of a given closed ideals J of A are in one-to-one correspondence to l.s.c. dimension functions D on $\bigcup_n M_n(J_{\min})$, and are in one-to-one correspondence to general l.s.c. 2-quasi-traces on A, it follows that our definition of "A is traceless" in 2.22 is equivalent to [45, def. 4.2]. Moreover, A is traceless if and only if for every σ-unital hereditary C^*-subalgebra $B \subset A$ every bounded 2-quasi-trace on B_+ is zero: indeed, a l.s.c. 2-quasi-trace τ on A_+ takes a value $0 < \tau(a) < \infty$ at $a \in A_+$ if and only if for all sufficiently small $\delta \in (0, \|a\|)$ the restriction of τ to $\overline{(a-\delta)_+ A (a-\delta)_+}$ is bounded and non-zero.

Semi-finite lower semi-continuous 2-quasi-traces τ on A_+ are in one-to-one correspondence to lower semi-continuous dimension functions D on $A \otimes \mathcal{K}$. Every semi-finite l.s.c. 2-quasi-trace on the positive part of a full hereditary C^*-subalgebra of A extends uniquely to a semi-finite l.s.c. 2-quasi-trace on A_+. Clearly the restriction of a semi-finite 2-quasi-trace to a hereditary C^*-subalgebras is again a semi-finite 2-quasi-trace. In conjunction with [5] it follows that a simple C^*-algebra A is stably finite if and only if there exists a faithful semi-finite l.s.c. 2-quasi-trace on A.

Remarks 2.28. Here we list the key ideas for the non-trivial parts of the proof of Proposition 2.24. A detailed proof can be found in [41].

(i) If D is is a local rank function on A, $a, b \in A_+$, and there is $\kappa > 0$ such that $\kappa D(f+g) \leq D(f) + D(g)$ for $f, g \in C^*(a,b)_+$, then

$$\kappa^{1/2} \tau_D(a+b)^{1/2} \leq \tau_D(a)^{1/2} + \tau_D(b)^{1/2}.$$

(Use (8) and (1) to get $\kappa D((a+b-t)_+) \leq D((a - t/(1+x))_+) + D((b - xt/(1+x))_+)$ for $x, t > 0$. Transformations in (2.9) show $\kappa \tau_D(a+b) \leq (1+x)\tau_D(a) + (1+1/x)\tau_D(b)$.)

(ii) Local rank functions on C^*-algebras A of real rank zero are determined by the values on its projections and are sub-additive rank functions: $D(a) = \sup\{D(p);\ p \in \mathrm{Proj}(\overline{a^*Aa})\}$. Conversely every function D on the projections in A with values in $[0, \infty]$ such that $D(p) = D(q)$ if $p \sim q$ (Murray–von Neumann equivalence) and $D(p + q) = D(p) + D(q)$ for projections $p, q \in A$ with $pq = 0$ determines a sub-additive rank function D on A in the sense of Definition 2.23.

If A has moreover stable rank one then for every local rank function D on A there is a unique sub-additive rank function D_n on $M_n(A)$ with $D_n(a \otimes e_{1,1}) = D(a)$, and every locally lower semi-continuous local quasi-trace τ on A is a lower semi-continuous 2-quasi-trace.

(iii) For a free ultrafilter ω on $\mathbb{N}$ we define a lower semi-continuous local quasi-trace τ_ω on $\ell_\infty(A)_+$ for $a = (a_1, a_2, \dots) \geq 0$ by

$$\tau_\omega(a) := \sup_{t>0} \lim_{\omega} \tau((a_n - t)_+).$$

(iv) Let $(C_1, \rho_1), (C_2, \rho_2), \dots$ be a sequence of commutative C^*-algebras C_n with positive functionals ρ_n on C_n such that $\gamma := \sup \|\rho_n\| < \infty$ and let ω be a free ultrafilter on $\mathbb{N}$. One defines a positive functional ρ_ω on $\ell_\infty\{C_n\}$ by $\rho_\omega(a_1, a_2, \dots) := \lim_\omega \rho_n(a_n)$.

The ultrapower $\prod_\omega (C_n, \rho_n)$ of $C_1, C_2, \dots$ with respect to $\rho_1, \rho_2, \dots$ is defined by $\ell_\infty\{C_n\}/I$ where $I := \{a \in \ell_\infty\{C_n\};\ \rho_\omega(a^*a) = 0\}$. It is folklore that $\prod_\omega (C_n, \rho_n)$ is a von Neumann algebra with faithful positive normal functional $[\rho_\omega](a + I) := \rho_\omega(a)$, e.g. it is a special case of [35, prop. 2.1].

(v) Suppose that τ is a faithful locally lower semi-continuous local quasi-trace B_+ such that for $a, b \in B_+$ with $ab = a$ and $\|b\| \leq 1$ there is a *-monomorphism $\varphi \colon N \to B$ from a commutative von Neumann algebra N into B such that $a \in \varphi(N)$, $\tau(\varphi(1)) < \infty$ and $\tau\varphi \colon N_+ \to [0, \infty)$ extends to a normal positive functional on N. Then B has real rank zero and stable rank one, pBp is a finite AW*-algebra for every projection $p \in B$, and τ is a faithful semi-finite lower semi-continuous 2-quasi-trace on B_+.

(vi) Suppose that τ is a l.s.c. local quasi-trace on A_+, that J a closed ideal of A with $J \subset I_\tau$ and that p is a projection in A/J. Then $\tau(a) = \tau(b)$ if $a, b \in A_+$ and $a + J = p = b + J$:
It suffices to consider the restriction of τ to $C^*(a, b)_+$. Suppose $A = C^*(a, b)$, then $A/J \cong \mathbb{C}.p$ and by Lemma 2.14 there is a contraction $c \in A_+$ such that $c + J = p$ and $c - c^2$ is a strictly positive element of J. Since $g_{1/2}(a) - a \in J \cap C^*(a)$ and $\tau(g_{1/2}(a)) = \tau(a)$, we can suppose that a is a contraction. It follows $\tau((c^\alpha - \delta)_+) = (1 - \delta)\tau(c)$ for $\delta \in [0, 1)$ and all $\alpha > 0$, , $\lim \|c^{1/n} a c^{1/n} - a\| = 0$ and $\lim \|c^n a c^n - c^{2n}\| = 0$. Thus, $\tau(a) = \tau(c) = \tau(b)$.

(vii) Suppose that J is a closed ideal of A and A/J has real rank zero. If τ is a semi-finite l.s.c. local quasi-trace on A_+ with $J \subset I_\tau$ then there is a semi-finite l.s.c. quasi-trace ρ on A/J such that $\rho(a + J) = \tau(a)$ for $a \in A_+$. (cf. (vi) and (ii).)

(viii) τ_ω is always a local quasi-trace on $\ell_\infty(A)_+$. I_{τ_ω} is an ideal of $\ell_\infty(A)$ if and only if $\inf_{t>0} Q(\tau, t) = 0$. If $\inf_{t>0} Q(\tau, t) = 0$ then I_τ and I_{τ_ω} are closed ideals, $\mathrm{Dom}_{1/2}(\tau_\omega)$ and $\mathrm{Dom}_{1/2}(\tau)$ are *-ideals.

(ix) If $I := I_{\tau_\omega}$ is an ideal then $J := \overline{\mathrm{Dom}_{1/2}(\tau_\omega)}$ is a closed ideal of $\ell_\infty(A)$. $B := J/I$ has real rank zero and there is a semi-finite l.s.c. quasi-trace ρ on B_+ such that $\rho(a+I) = \tau(a)$ for $a \in A_+$. Thus, $\tau_\omega(a) = \tau_\omega(b)$ if $a, b \in \ell_\infty(A)_+$ and $b - a \in I_+$. For every pair of bounded sequences $a = (a_1, a_2, \ldots), b \in \ell_\infty(A)_+$ with $a_n b_n = a_n$, $\|b_n\| \le 1$ and $\sup_n\{\tau((b_n - t)_+)\} < \infty$ for every $t > 0$ the quotient $N := C/(I \cap C)$ of $C := \ell_\infty\{C^*(a_n)\} \subset \ell_\infty(A)$ is a von Neumann algebra and the class map $[\tau_\omega]\colon N_+ \to [0, \infty)$ extends to a faithful normal positive linear functional f on N with $f(c) = \rho(c)$ for $c \in C_+$. Hence pBp is a finite AW*-algebra for every projection $p \in B$.

Remarks 2.29. (i) A theorem of U. Haagerup [27] says that every bounded 2-quasi-trace on a unital exact C^*-algebra A extends to a (linear) trace on A. One gets that *all lower semi-continuous 2-quasi-traces on (not necessarily unital) exact C^*-algebras A_+ are additive traces on A_+* by elementary reductions to the unital case as in Remark 2.27(vii).

(ii) It follows from obvious modifications of the proofs of [27, lemma 5.7], [27, lemma 5.8] and from Remark 2.28(iii) that for every bounded 2-quasi-trace τ on A there is a bounded 2-quasi-traces τ_0, on $(A \otimes \ell_\infty\{M_{2^n};\, n \in \mathbb{N}\})_+$ such that $\tau_0(a \otimes b) = \tau(a) \lim_\omega \mathrm{tr}(b_n)$ for $a \in A_+$ and $b = (b_1, b_2, \ldots) \in \ell_\infty\{M_{2^n}\}_+$. Thus, the kernel-ideal I_{τ_0} contains $A \otimes I_\omega$ and τ_0 defines a bounded 2-quasi-trace $[\tau_0]$ on $A \otimes \ell_\infty\{M_{2^n}\}/A \otimes I_\omega$, where $I_\omega := \{(b_1, b_2, \ldots) \in \ell_\infty\{M_{2^n}\}\,;\ \lim_\omega \mathrm{tr}(b_n^* b_n) = 0\}$.
Since the CAR-algebra M_{2^∞} is nuclear and is a C^*-subalgebra of $\ell_\infty\{M_{2^n}\}/I_\omega$, $[\tau_0]$ induces a bounded 2-quasi-trace τ_1 on $(A \otimes M_{2^\infty})_+$ such that $\tau_1(a \otimes b) = \tau(a)\mathrm{tr}(b)$.

By the obvious modification of [27, lemma 5.8] and in conjunction with the up-down theorem (for weakly dense C^*-subalgebras of von-Neumann algebras with separable preduals), it follows that there is a bounded 2-quasi-trace τ_2 on $(A \otimes \mathcal{R})_+$ with $\rho_1(a \otimes b) = \rho(a)\mathrm{tr}(b)$ for $a \in D_+$ and $b \in \mathcal{R}_+$, where $\mathcal{R}$ denotes the hyperfinite II_1 factor with separable predual.

(iii) Is I_τ a vector space for every "trivial" lower semi-continuous quasi-trace τ on A_+ for every extension $0 \to C_0((0,1], \mathcal{K}) \to A \to C([0,1] \times [0,1]) \to 0$? It is not known if a stably infinite simple nuclear C^*-algebra A can admit a non-zero bounded quasi-trace on A_+: it can not be a 2-quasi-trace.

If τ is a l.s.c. quasi-trace then a trivial l.s.c. quasi-trace $\infty \cdot \tau$ is given by $(\infty \cdot \tau)(a) := 0$ if $\tau(a) = 0$ and $(\infty \cdot \tau)(a) := \infty$ if $\tau(a) \ne 0$, i.e., $\infty \cdot \tau(a) = \sup_{t>0} t\tau(a)$. The map $\infty \cdot \tau$ satisfies $I_\tau = \mathrm{Dom}_{1/2}(\infty \cdot \tau) = I_{\infty \cdot \tau}$.

For arbitrary quasi-traces τ one can define a trivial l.s.c. quasitrace τ_0 as follows: let $\tau_0(a) = 0$ if $\tau((a - \delta)_+) < \infty$ for every $\delta > 0$ and let $\tau_0(a) = \infty$ otherwise. The set I_{τ_0} is the closure of $\mathrm{Dom}_{1/2}(\tau)$ if τ is l.s.c.

If we consider $\infty \cdot \tau$, then Proposition 2.24 and Remark 2.28(viii) show that every l.s.c. quasi-trace is a 2-quasi-trace if the kernel I_τ is an ideal for every trivial l.s.c. quasi-trace τ.

The kernels I_τ of trivial l.s.c. quasi-traces are just all closed sets $X \subset A$ with $aXb \subset X$ for all $a, b \in A$ such that the intersections $X \cap C$ with commutative C^*-subalgebras of A are linear subspaces of C. It was a question whether those X are ideals of A, but the answer is negative by Proposition 2.25.

(iv) Let $\mathcal{A}_n$ $(n = 2, 3, \dots)$ denote the universal unital C^*-algebra with generators $a_1, \dots, a_n, b$ and defining relations $\sum_k a_k(a_k)^* = 2$ and $b^*b + \sum_k (a_k)^* a_k = 1$. The logical sum of [27, lem. 2.1] and [44, prop. 5.7] implies that *every bounded* 2*-quasi-trace is a trace if and only if the unity of* $\mathcal{A}_n \otimes M_2 \otimes M_3 \otimes M_4 \otimes \dots$ *is properly infinite for every* $n > 2$. Clearly $\mathcal{A}_2 = \mathcal{E}_2$. The $\mathcal{A}_n$ are in a weak sense almost semi-projective.

(v) Even finite local quasi-traces on $C([0,1] \times [0,1])$ are not quasi-traces: the bounded local quasi-traces τ on commutative unital C^*-algebras $A \cong C(X)$ (which we can normalize such that $\tau(1) = 1$) are restrictions to A_+ of quasi-states in the sense of [1] on A. By [1] there is a non-linear quasi-state τ_A on $B := C([0,1] \times [0,1])$. An examination of the ideas in Remarks 2.28 shows that for $\tau := \tau_A | B_+$ and for the corresponding τ_ω the set I_{τ_ω} is not an ideal of $\ell_\infty(B)$, i.e., $\inf_{t>0} Q(\tau, t) > 0$. It follows also that there is a closed ideal J of $\ell_\infty(B)$ which is contained in the set I_{τ_ω} and elements $a, b \in \ell_\infty(B)_+$ such that $a - b \in J$ but $\tau_\omega(a) \neq \tau_\omega(b)$. Similar considerations happen for the bounded quasi-trace $\tau_A \pi$ on $C([0,1]) * C([0,1])_+$ considered in Proposition 2.25.

Lemma 2.30. *Suppose that a lower semi-continuous quasi-trace* $\tau \colon A \to [0, \infty]$ *is finite on the positive part of a stable* C^**-subalgebra* B *of* A *(i.e.,* $\tau(b) \in [0, +\infty)$ *for every* $b \in B_+$*).*

Then $\tau(a) = 0$ *for every* $a \geq 0$ *in the closed ideal of* A *which is generated by* B*.*

Proof. Since $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_\infty \cong C^*(s_1, s_2, \dots)$ unitally, we have $\sum_{1 \leq k \leq n} \tau(s_k a s_k^*)/k \leq \tau(\sum_m s_m a s_m^*/m) < \infty$ for every $n > 0$ if $a \in B_+$. The divergence of the harmonic series implies $\tau(a) = 0$ because $\tau((s_k a^{1/2})(s_k a^{1/2})^*) = \tau(a)$.

Let R denote the closed linear span of BA. The stability of B implies that every positive element a of $J = \overline{\operatorname{span}(ABA)}$ is of the form $a = c^*c$ with c in R. By the Cohen factorization theorem, the non-degenerate B-module R is just the set of products BA itself; thus, $cc^* \leq b$ for some $b \in B_+$. A lower semi-continuous quasi-trace is monotone, cf. Remarks 2.27(iv). Thus, $\tau(a) = \tau(cc^*) \leq \tau(b) = 0$ for $a \in J_+$. □

3. Locally purely infinite tensor products and simple C^*-algebras

The notion of purely infinite C^*-algebra was introduced by J. Cuntz on p.186 of [17]. He defines a (simple) C^*-algebra to be *purely infinite* if every non-zero hereditary C^*-subalgebra contains an infinite projection, i.e., a projection which is Murray–von Neumann equivalent to a proper subprojection of itself. This is equivalent to our Definition 1.1 in the case of simple algebras. This is well-known, but we add here a self-contained proof of it and show that in the case of simple C^*-algebras the property l.p.i. is equivalent to property p.i.

Further, we study the question when $A \otimes B$ is locally purely infinite. We apply the main result to tensorially non-prime C^*-algebras and tensor products with $C_r^*(F_2)$.

Proposition 3.1. *Let* A *be a non-zero* C^**-algebra. The following statements are equivalent.*

(i) *A is simple and every non-zero hereditary C^*-subalgebra of A contains an infinite projection.*

(ii) *$A \neq \mathbb{C}$, and for $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon > 0$, there exists $c \in A$ with $\|c\| = 1$ such that $\|b - c^*ac\| < \varepsilon$.*

(iii) *A is simple and locally purely infinite (i.e., A is simple and every non-zero hereditary C^*-subalgebra of A contains a non-zero stable C^*-subalgebra).*

Note that the infinite projection p in (i) must also be properly infinite by (ii), i.e., if s is a partial isometry with $ss^* < s^*s = p$, then there exists $t \in A$ with $t^*t = p$ and $tt^* \leq p - ss^*$ (which implies $t^*s = 0$). More generally every infinite projection in a *simple* C^*-algebra is properly infinite, cf. Cuntz [15].

For non-simple C^*-algebras the equivalence of (i) and (ii) does not hold: The C^*-algebra $C((0,1]; \mathcal{O}_2)$ is purely infinite in the sense of Definition 1.1 but it contains no projection, whereas the unitization of $\mathcal{O}_2 \otimes \mathcal{K}$ is purely infinite in the sense of Cuntz and does not satisfy the criteria of Definition 1.1 or of Definition 1.3.

Proof. (i)⇒(iii) Let $D \subset A$ be a non-zero hereditary C^*-subalgebra. Then it contains a C^*-subalgebra which is isomorphic to the Toeplitz algebra $C^*(s\colon\ s^*s = 1)$. Thus, D also contains an isomorphic copy of $\mathcal{K}$.

(iii)⇒(ii): First of all, $A \neq \mathbb{C}$, because $\mathbb{C}$ does not contain any non-zero stable C^*-subalgebra. Let $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $1 > \varepsilon > 0$ be given. Put $\eta := \varepsilon/3$. Then $e = (a - 1 + \eta)_+ \in A_+$ is non-zero and there are by assumption elements $f_1, f_2, \ldots$ in the hereditary C^*-subalgebra $D := \overline{eAe}$ with $f_i^* f_j = \delta_{i,j} d$ for some non-zero $d \in D_+$. Since A is simple, one can find $g_1, \ldots, g_n \in A$ with $\|b - \sum g_k^* d g_k\| < \eta$. Let $h := \sum_{1 \leq k \leq n} f_k g_k$. Then $\|h^*h - b\| < \eta$ and thus $0 < 1 - \eta < \|h\|^2 < 1 + \eta$. On the other hand $(a - e)h = (1 - \eta)h$, because $h \in D$ and $(1 - \eta)^{-1}(a - e)e = e$. Therefore the element $c := h/\|h\|$ satisfies $\|c\| = 1$ and

$$\|c^*ac - b\| \leq \|a - (1 - \eta)^{-1}(a - e)\| + |1 - \|h\|^2| + \|h^*h - b\| < 3\eta = \varepsilon\,.$$

(ii)⇒(i): The compact operators on a Hilbert space H (of dimension > 1) do not satisfy the criteria listed under (ii), because a one-dimensional projection is not equivalent to a two-dimensional projection. The properties of A imply that A is simple. If p is a non-zero projection, then the unital C^*-subalgebra $D := pAp$ contains a non-zero element $b \in D_+$ with $0 \in \mathrm{Spec}_D(b)$, because A is not isomorphic to the compact operators on a Hilbert space. By assumption we find $c \in A$ with $\|c^*b^2c - p\| < 1/2$. Then $bcp \in pAp$ is left-invertible but is not right-invertible in pAp. This shows that every non-zero projection $p \in A$ is infinite.

It suffices now to prove that every non-zero hereditary C^*-subalgebra E of A contains a non-zero projection. Take $a \in E_+$ with $\|a\| = 1$. Choose contractions $c_n \in A$ with $\|c_n^* a^{2n} c_n - a^{1/n}\| \leq 1/n$. One can then define a contraction z in $\ell_\infty(E)/c_0(E)$ by the representing sequence $z_n = a^n c_n a^{1/n} \in E$. If one embeds E naturally in $\ell_\infty(E)/c_0(E)$ as constant sequences, then $z^*za = a$, $az = z$. It entails that $(1 - z^*z)^{1/2}z = 0$, $(a - a^2)(1 - z^*z)^{1/2} = 0$ and $(a - a^2)z = 0$. Thus, $w = z + (1 - z^*z)^{1/2}$ is an isometry in the unitization of $\ell_\infty(E)/c_0(E)$ with $w^*(a - a^2)w = 0$ and $1 - ww^* \in \ell_\infty(E)/c_0(E)$. It follows that at least one of a or $1 - ww^*$ must be a non-zero projection in $\ell_\infty(E)/c_0(E)$. But by functional calculus, a non-zero projection in $\ell_\infty(E)/c_0(E)$ can be represented by a sequence of projections $p_n \in E$, where at least one of the projections is non-zero. □

The trick with w in the last part of the above proof is quoted to M. Rieffel, but it was also used by B. Blackadar and J. Cuntz [5] to show that stable simple C^*-algebras without any non trivial lower semi-continuous dimension function contain a (properly) infinite projection.

It was for a long time an open problem whether stably infinite simple C^*-algebras are purely infinite, but M. Rørdam [59] constructed a nuclear, simple and stable C^*-algebra which contains both infinite and non-zero finite projections. Thus, the absence of non-trivial lower semi-continuous dimension functions on a stable C^*-algebra A does not imply that A is locally purely infinite.

A dichotomy between existence of non-trivial lower semi-continuous traces and pure infiniteness is established in Corollary 3.11 for certain exact simple algebras. Before we study the permanence of l.p.i. with respect to tensor products. This is non-trivial because in general $A \otimes B$ is not locally purely infinite if B is strongly purely infinite, cf. [40].

Remark 3.2. We say that A is *strictly anti-liminal* if every quotient of A is anti-liminal. Equivalently this means that the image of every irreducible representation of A has zero intersection with the compact operators. It follows that every non-zero hereditary C^*-subalgebra D of a strictly anti-liminal C^*-algebra A is again strictly anti-liminal.

Lemma 3.3. *Let φ be a pure state on a strictly anti-liminal C^*-algebra A and $a \in A_+$ be a non-zero positive element with $\varphi(a) = \|a\|$.*

Then for every $n \in \mathbb{N}$ there exists a morphism $\lambda\colon E := C_0((0,1], M_n) \to \overline{aAa}$ such that for $f_2 := \lambda(h_0 \otimes e_{21})$ the restriction of φ to $\overline{\lambda(E)A\lambda(E)} \cong \overline{f_2^*Af_2} \otimes M_n$ is non-zero and is (up to isomorphism) of form $(\varphi|\overline{f_2^*Af_2}) \otimes \rho_0$, where $\rho_0\colon [\alpha_{i,j}] \mapsto \alpha_{1,1}$.

In particular, $\varphi(f_2^*f_2) > 0$.

Recall that h_0 was defined in subsection 2.3.

Proof. Since $0 < \|a\| = \varphi(a)$, the restriction ψ of φ to $D := \overline{aAa}$ is a pure state. The irreducible cyclic representation $d\colon D \to \mathcal{L}(\mathcal{H})$ with cyclic vector ξ corresponding to ψ is of infinite dimension, because D is strictly anti-liminal. Let $\xi_1 := \xi, \ldots, \xi_n$ be n ortho-normal elements of $\mathcal{H}$ and let $I\colon \mathbb{C}^n \to \mathcal{H}$ the isometry defined by them. As noted at the end of subsection 2.3 there is a morphism $\lambda\colon C_0((0,1], M_n) \to D$ such that $\langle d(\lambda(f))\xi_j, \xi_k\rangle = f(1)_{jk}$ for $f \in C_0((0,1], M_n)$. Under the natural isomorphism

$$\overline{\lambda(C_0((0,1], M_n))A\lambda(C_0((0,1], M_n))} \cong \overline{f_2^*Af_2} \otimes M_n$$

the restriction of φ becomes $(\varphi|\overline{f_2^*Af_2}) \otimes \rho_0$. □

Remark 3.4. Let $c = v(c^*c)^{1/2} = (cc^*)^{1/2}v$ be the polar decomposition of $c \in A$ in the enveloping von Neumann algebra A^{**}, then the map $d \mapsto vdv^*$ defines an isomorphism from the hereditary C^*-subalgebra generated by c^*c onto the hereditary C^*-subalgebra generated by cc^* (cf. e.g. [44, lemma 2.4]).

Theorem 3.5. *Suppose that A and B satisfy the following conditions (i)–(iii); then the spatial C^*-algebra tensor product $A \otimes B$ is locally purely infinite.*

(i) *The natural map from* $\operatorname{prime}(A) \times \operatorname{prime}(B)$ *into* $\operatorname{prime}(A \otimes B)$ *is an isomorphism.*

(ii) A *is strictly anti-liminal, i.e., every irreducible representation of* A *has zero intersection with the compact operators.*

(iii) *For every primitive ideal* J *of* B *and* $b \in B_+ \setminus J$ *there exists* $n \in \mathbb{N}$ *such that for every primitive ideal* I *of* A *and for every non-zero positive element* $a \in A_+ \setminus I$ *there exists a stable* C^**-subalgebra* $C \subset \overline{aAa} \otimes M_n \otimes \overline{bBb}$ *which is not contained in* $(I \otimes M_n \otimes B) + (A \otimes M_n \otimes J)$*.*

Proof. Let K be a primitive ideal of $A \otimes B$ and $d \in (A \otimes B)_+ \setminus K$. Let $D := \overline{d(A \otimes B)d}$. By (i) and Lemma 2.18 there are non-zero $a_0 \in A_+$, $b \in B_+$, $t \in A \otimes B$ and pure states φ on A and ψ on B such that $(\varphi \otimes \psi)(K) = 0$, $tt^* \in D$, $t^*t = a_0 \otimes b$, $\varphi(a_0) = \|a_0\| = 1$ and $\psi(b) = \|b\| = 1$. Let $I \triangleleft A$ and $J \triangleleft B$ be the kernels of the irreducible representations corresponding to the irreducible representations ρ_1 and ρ_2 defined by the pure states φ respectively ψ. The kernel of the irreducible representation $\rho_1 \otimes \rho_2$ corresponds to the pure state $\varphi \otimes \psi$ on $A \otimes B$ and is $I \otimes B + A \otimes J$ by assumption (i) and Proposition 2.16. Thus, $K \subset I \otimes B + A \otimes J$. Remark 3.4 implies that D contains a stable C^*-subalgebra which is not contained in K if there is a stable C^*-subalgebra C of $\overline{a_0 A a_0} \otimes \overline{bBb}$ which is not contained in $I \otimes B + A \otimes J$.

Let $n = n(b, J) \in \mathbb{N}$ be as in assumption (iii) and let f_2 be as in Lemma 3.3. Then for $a := f_2^* f_2$ we have $\varphi(a) > 0$, $\overline{aAa} = \overline{b_2^* A b_2}$, and there is a *-monomorphism $h \colon \overline{aAa} \otimes M_n \hookrightarrow \overline{a_1 A a_1}$ such that $h(x \otimes e_{1,1}) = x$ for $x \in \overline{aAa}$ and $\varphi h = (\varphi | \overline{aAa}) \otimes \rho_0$. It follows $I \cap h(\overline{aAa} \otimes M_n) = h((I \cap \overline{aAa}) \otimes M_n)$. By (iii) there is a stable (w.l.o.g.) hereditary C^*-subalgebra C of $\overline{aAa} \otimes M_n \otimes \overline{bBb}$ which is not contained in $(I \otimes M_n \otimes B) + (A \otimes M_n \otimes J)$ because a is not in I. It follows that the stable hereditary C^*-subalgebra $F := h \otimes \mathrm{id}_B(C)$ of $\overline{a_1 A a_1} \otimes \overline{bBb}$ is not contained in $I \otimes B + A \otimes J$, because there is $e \in \overline{aAa} \otimes M_n \otimes \overline{bBb}$ with $ee^* \in C$ and $(\varphi \otimes \rho_0 \otimes \psi)(e^*e) > 0$, i.e., $ff^* \in E$ and $(\varphi \otimes \psi)(f^*f) > 0$ for $f := (h \otimes \mathrm{id})(e)$. $\square$

Remark 3.6. One can reformulate (iii) as follows with help of condition (i), Proposition 2.16 and Lemma 2.18:

(iii') For every non-zero positive element $b \in B_+$ and every pure state ψ on B with $\psi(b) > 0$ there is an $n \in \mathbb{N}$ such that for every non-zero positive element $a \in A_+$ and every pure state φ on A with $\varphi(a) > 0$ there exists a stable C^*-subalgebra $C \subset \overline{aAa} \otimes M_n \otimes \overline{bBb}$ such that the restriction of $\varphi \otimes \mathrm{Tr}_n \otimes \psi$ to the ideal generated by C is non-zero.

Remark 3.7. Recall that a simple C^*-algebra A is *stably infinite* if there is an $n \in \mathbb{N}$ such that $M_n(A)$ contains an infinite projection.

Simple (!) A is called *stably finite* if A is not stably infinite.

Note that a simple C^*-algebra A is stably infinite if and only if it has no faithful semi-finite lower semi-continuous 2-quasi-trace (cf. Definition 2.22) by [5]. This is equivalent to the absence of faithful semi-finite lower semi-continuous traces if A is exact, cf. 2.29(i).

Definition 3.8. We call a C^*-algebra A *weakly stably infinite* if for every primitive ideal I of A and for every non-zero positive $a \in A_+ \setminus I$ there exist $n \in \mathbb{N}$ and a stable C^*-subalgebra C in $\overline{aAa} \otimes M_n$ which is not contained in $I \otimes M_n$.

(The negation of this property for a non-simple C^*-algebra A is not useful and should not be called "weakly stably finite", but A could be called *residually stably finite* if for every primitive ideal I of A the algebra $(A/I) \otimes M_2 \otimes M_3 \otimes \ldots$ does not contain a non-zero stable C^*-subalgebra.)

A simple C^*-algebras A is weakly stably infinite if and only if A is stably infinite:

If A simple and q is an infinite projection in $M_m \otimes A$ then for every non-zero $a \in A_+$ there exists $n \in \mathbb{N}$ with $n \geq m$ and a partial isometry $u \in M_n \otimes A$ with $uu^* = q$ and $u^*u \in M_n \otimes \overline{aAa}$. The projection $p := u^*u$ is infinite. Let $v \in M_n \otimes \overline{aAa}$ be a partial isometry with $v^*v \leq v^*v = p$ and $p \neq v^*v$. Then $C^*(v)$ is isomorphic to the Toeplitz algebra which contains $\mathcal{K}$. Thus, A is weakly stably infinite.

Conversely if A is stably finite then there is a faithful semi-finite lower semi-continuous 2-quasi-trace τ on A_+. Let $a \in A_+$ with $0 < \tau(a) < \infty$, $t := \|a\|/2$ and let $D := \overline{(a-t)_+ A (a-t)_+}$. The restriction of τ to D_+ is bounded and faithful. For every $n \in \mathbb{N}$ there is a bounded 2-quasi-trace τ_n on $(D \otimes M_n)_n$ with $\tau(d) = \tau_n(d \otimes e_{1,1})$ for $d \in D_+$. Since $D \otimes M_n$ is simple, there is no non-zero stable C^*-subalgebra in $D \otimes M_n$ by Lemma 2.30. Thus, A is not weakly stably infinite.

Corollary 3.9. *In the following cases the spatial tensor product $A \otimes B$ is locally purely infinite:*

(i) *B is weakly stably infinite (cf. Def. 3.8),* $\mathrm{prime}(A) \times \mathrm{prime}(B) \cong \mathrm{prime}(A \otimes B)$ *(naturally) and A is strictly anti-liminal.*

(ii) *B is simple and stably infinite (i.e., is not stably finite), A strictly anti-liminal and*
$$I \otimes B \to A \otimes B \to (A/I) \otimes B$$
is exact for every closed ideal I of A.

(iii) *A is locally reflexive and strictly anti-liminal and B is simple and stably infinite.*

(iv) *A or B is exact, A is strictly anti-liminal and B is simple and stably infinite.*

(v) *A is locally purely infinite and* $\mathrm{prime}(A \otimes B)$ *is naturally isomorphic to* $\mathrm{prime}(A) \times \mathrm{prime}(B)$.

(vi) *A is locally purely infinite and A or B is exact.*

(vii) *A is locally purely infinite and locally reflexive and B is simple.*

(viii) *$B \cong C^*_r(F_2)$ and zero is the only bounded positive (linear) trace on D for every hereditary C^*-subalgebra D of A.*

Proof. Part (i) is a special case of of Theorem 3.5: the condition (iii) of Theorem 3.5 is satisfied if B is locally stably infinite, because $C := C^*(a) \otimes D$ is stable and the ideal generated by it is not contained in $I \otimes M_n \otimes B + A \otimes M_n \otimes J$ if $a \in A_+$ is not in I and if D is a stable C^*-subalgebra of $M_n \otimes \overline{bBb}$ which not contained in $M_n \otimes J$. Indeed, if $d \in D_+ \setminus M_n \otimes J$ then $a \otimes d$ is not in $I \otimes M_n \otimes B + A \otimes M_n \otimes J$.

In all cases (ii)-(viii) the condition (i) of Theorem 3.5 is satisfied by Proposition 2.17. In particular, (vi) and (vii) are special cases of (v).

A satisfies assumption (ii) of Theorem 3.5, i.e., A is strictly anti-liminal, if A is locally purely infinite or if every hereditary C^*-subalgebra D of A has only zero as bounded (linear) trace: if I is a primitive ideal of A and $b \in A_+$ such that $\pi_I(bAb)$ is of finite dimension, then $b \in I$, because otherwise $D := \overline{bAb}$ can not contain a stable C^*-subalgebra which is not contained in I and D admits a non-zero bounded linear trace.

(v): Condition (iii) of Theorem 3.5 is satisfied with $n = 1$, because $C := D \otimes C^*(b)$ is stable and not contained in $I \otimes \otimes B + A \otimes J$ if $b \in B_+$ is not in J and if D is a stable C^*-subalgebra of $\overline{aAa}$ which not contained in I. Above we have observed condition (ii) of Theorem 3.5.

(ii)-(iv) are special cases of (v): If B is a simple C^*-algebra which is not stably finite, is stably infinite and thus weakly stably infinite.

(viii): As we have seen above, A and $B \cong C^*_r(F_2)$ satisfy conditions (i) and (ii) of Theorem 3.5. Since B is simple, for every non-zero positive $b \in B_+$ there is $n \in \mathbb{N}$ such that in $M_n \otimes B$ there is a partial isometry v with $v^*v = e_{1,1} \otimes 1$ and $vv^* \in M_n \otimes \overline{bBb}$. If $a \in A_+$ and $\delta > 0$ then the elements $(a \otimes vv^* - \delta)_+$ are equivalent to $(a - \delta)_+ \otimes e_{1,1} \otimes 1$ in $A \otimes B$ by $1 \otimes v \in \mathcal{M}(A \otimes B)$. The elements $(a - \delta)_+ \otimes e_{1,1} \otimes 1$ are zero or properly infinite in $\overline{(aAa)} \otimes e_{1,1} \otimes B$ by Lemma 2.21. Thus, the positive element $a \otimes vv^*$ in $\overline{aAa} \otimes M_n \otimes \overline{bBb}$ satisfies the assumption of Lemma 2.10. It follows that for every $\varepsilon > 0$ there is a stable C^*-subalgebra C of $\overline{aAa} \otimes M_n \otimes \overline{bBb}$ which contains $((a \otimes vv^*) - \varepsilon)_+$ in the ideal generated by C. This implies condition (iii) of Theorem 3.5. $\square$

Definition 3.10. A simple C^*-algebra A is *tensorially non-prime* if A is isomorphic to the tensor product $B \otimes C$ of two simple C^*-algebras B and C which are both not isomorphic to the compact operators on a Hilbert space.

Corollary 3.11. *Suppose that A and B are simple C^*-algebras, which both are not isomorphic to the compact operators on a Hilbert space.*

(i) *If A or B is stably infinite, then the spatial tensor product $A \otimes B$ is purely infinite (and simple).*
(ii) *If $A \otimes B$ is exact and stably infinite, then $A \otimes B$ is purely infinite.*
(iii) *If A has no faithful semi-finite lower semi-continuous (additive) trace and $B \cong C^*_r(F_2)$, then $A \otimes B$ is purely infinite.*
(iv) *A is (quasi-)traceless if $A \otimes B$ is purely infinite and B is nuclear and stably finite.*

In particular, $A \otimes B$ is p.i. if B is p.i. and A is simple, e.g. $A \otimes \mathcal{O}_n$ is purely infinite for $n = 2, \ldots, \infty$ and every simple A. (Note that $\mathcal{K}(H) \otimes B$ is purely infinite if B is purely infinite, because pi(1) is a stable property, cf. e.g. [44, thm. 4.23].)

Part (iii) shows that (iv) holds for every exact B if and only if we could replace in (iv) B by $C^*_r(F_2)$ then all lower semi-continuous 2-quasi-traces are (additive) traces.

The below given proof shows also: *An exact simple tensorially non-prime C^*-algebra has a faithful semi-finite lower semi-continuous trace if and only if it is not purely infinite.*

Proof. (i): $A \otimes B$ is simple, cf. [61]. By symmetry, we can assume e.g. that B is not stably finite. Since A is simple and not of type I, A is strictly anti-liminal. Thus, the simple algebra $A \otimes B$ is locally purely infinite by part (ii) of Corollary 3.9 and is purely infinite by part (iii) of Proposition 3.1.

(ii): The C^*-algebras A and B are simple and exact, because $A \otimes B$ is simple and e.g. for $0 \neq b \in B_+$ $A \otimes C^*(b) \cong C_0(\mathrm{Spec}(b) \setminus \{0\}, A)$ and hence A is exact by the permanence properties of exactness, cf. [35].

Suppose that $A \otimes B$ is not purely infinite. Then A and B both are stably finite by part (i). By Remarks 2.27(viii) and 2.29(i) there are faithful lower semi-continuous semi-finite additive traces on A_+ and B_+. Thus, there is a faithful lower semi-continuous trace on $(A \otimes B)_+$, which contradicts that $A \otimes B$ is stably infinite.

(iii): Since A is simple, A has a faithful lower semi-continuous semi-finite additive trace if an only if there exists a hereditary C^*-subalgebra D of A and a tracial state on D. Thus, (iii) follows from part (viii) of Corollary 3.9 and part (iii) of Proposition 3.1.

(iv): Suppose that A has non-trivial lower semi-continuous 2-quasi-trace ρ. Then there exist $a \in A_+$ with $0 < \rho(a) < \infty$. Since ρ is lower semi-continuous, it follows that there exists $\delta > 0$ such that also $0 < \rho((a - \delta)_+)$. It follows that the restriction of ρ to $D := \overline{(a - \delta)_+ A (a - \delta)_+}$ is a non-zero finite 2-quasi-trace on D. By Remark 2.29(ii) there is a bounded 2-quasi-trace ρ_1 on $(D \otimes \mathcal{R})_+$ with $\rho_1(a \otimes b) = \rho(a)\mathrm{tr}(b)$ for $a \in D_+$ and $b \in \mathcal{R}_+$, where $\mathcal{R}$ denotes the hyperfinite II_1 factor with separable predual.

Since B is simple, nuclear and stably finite, there exists a faithful lower semi-continuous semi-finite (additive) trace τ on B_+ by [5] and [27], see Remarks 2.27(viii) and 2.29(i). Thus, there exists a non-zero hereditary C^*-subalgebra E of B such that $\tau|E$ is non-zero and finite. It follows that there is an extreme point τ_1 of the set of trace states on E. Since E is again nuclear and not of type I, there corresponds a *-monomorphism h from E into $\mathcal{R} \cong L(E)''$ such that $\mathrm{tr}(h(a)) = \tau_1(a)$ for $a \in E_+$. (Here $L \colon E \to \mathcal{L}(L_2(E, \tau_1))$ is given by the left-multiplication of elements of E on $L_2(E, \tau_1)$.)

The map $\rho_2 \colon c \mapsto \rho_1(\mathrm{id}_D \otimes h(c))$ is a bounded 2-quasi-trace on $(D \otimes E)_+$ with $\rho_2(a \otimes b) = \rho(a)\tau(b)$ for $a \in D_+$ and $b \in E_+$. In particular ρ_2 is non-zero and bounded on the positive part of the hereditary C^*-algebra $D \otimes E$ of $A \otimes B$, which contradicts the pure infiniteness of $A \otimes B$. □

4. Locally and weakly purely infinite non-simple C^*-algebras

We give an alternative characterization and some basic properties of l.p.i. algebras.

Proposition 4.1. (i) *A C^*-algebra A is locally purely infinite if and only if every hereditary C^*-subalgebra E of A is the closure of the union of finite sums of closed ideals of E which are generated by stable C^*-subalgebras.*

(ii) *Every l.p.i. C^*-algebra A is traceless (hence A is anti-liminal).*
(iii) *Every non-zero quotient and every non-zero hereditary C^*-subalgebra of a l.p.i. C^*-algebra is l.p.i.*

Proof. (i): The set of closed ideals of E which are finite direct sums of stably generated ideals clearly is upward directed. Thus, their closed union is an ideal I of E. (Here we consider 0 as stable algebra). Since E is hereditary in A, there is a closed ideal K of A such that $I = E \cap K$, e.g. let K be the closed linear span of AIA. If $I = E$ then every primitive ideal J of A which does not contain E can not contain all stable subalgebras of E. Thus, if $I = E$ for all hereditary C^*-subalgebras E of A, then A satisfies Definition 1.3 of l.p.i.

Conversely suppose $I \neq E$ for some hereditary C^*-subalgebra $E \subset A$. Then there exist $b \in E_+ \setminus K$ and, thus, a primitive ideal $J \supset K$ with $\|b + J\| > 0$. But J contains all stable C^*-subalgebras of E, i.e., A is not l.p.i.

(ii): Let $\tau\colon A_+ \to [0, \infty]$ be a lower semi-continuous 2-quasi-trace on A. Suppose there is $a \in A_+$ with $0 < \tau(a) < \infty$. Since τ is lower semi-continuous there is $\varepsilon > 0$ such that $e := (a - \varepsilon)_+$ satisfies $\tau(e) > 0$. Let $f := g_\delta(a)$ for $\delta := \varepsilon/2$ and g_δ as in formula (2.7). For every positive element c in the hereditary C^*-subalgebra $E := \overline{eAe}$ we have $cf = c = fc$, and, thus, $\tau(c) \leq \|c\|\tau(f)$. Since $\tau(f) \leq \tau(a)/\varepsilon < \infty$, we get that τ is bounded on E. By Lemma 2.30, τ is zero on every stably generated closed ideal of E. Since $I_\tau = \{d \in E;\ \tau(d^*d) = 0\}$ is a closed ideal of E for a l.s.c. 2-quasi-trace τ, we get from part (i) that $\tau(e) = 0$, a contradiction.

(iii): If I is a closed ideal, K a primitive ideal of A/I and $c \in (A/I)_+$ with $\|c+K\| > 0$, then there is $b \in A_+$ with $b + I = c$ and a primitive ideal J of A with $\pi_I(J) = K$, $I \subset J$. Thus, $\|b + J\| = \|c + K\| > 0$, and, by Definition 1.3, there is a stable hereditary C^*-subalgebra D of $\overline{bAb}$ which is not contained in J. Then $\pi_J(D)$ is stable and hereditary, is not contained in K, but is contained in $\overline{c(A/I)c}$.

Definition 1.3 passes to hereditary C^*-subalgebras E of A, because for every primitive ideal I of E there is a unique primitive ideal J of A with $I = J \cap E$. $\square$

Remarks 4.2. The property "l.p.i." is also a stable property. as a the special case of part (vi) of Corollary 3.9 with $B = \mathcal{K}$ shows.

Moreover, one can show that the class of l.p.i. C^*-algebras is closed under strong Morita equivalence and is preserved by inductive limits.

The C^*-algebra A is l.p.i. if and only if for every separable subset $X \subset A$ there is a separable C^*-subalgebra $B \subset A$ with B l.p.i. and $X \subset B$.

The converse implication of part (ii) of Proposition 4.1 does not hold, because there are stably infinite simple nuclear C^*-algebras which are not purely infinite, cf. [59]. One only has the following reformulation of the much weaker result [45, thm. 4.8] as a sort of "asymptotic" inverse.

Proposition 4.3. *Let A be a non-zero C^*-algebra and let A_ω be the ultra-power of A. Then the following assertions are equivalent.*

(i) *A_ω is traceless, i.e., every lower semi-continuous 2-quasi-trace on A_ω is trivial.*

(ii) *There exists $k \in \mathbb{N}$ such that $a \otimes 1_k$ is properly infinite for all $a \in A_+ \setminus \{0\}$.*

Proof. It is a reformulation of the equivalence of (a) and (c) in part (i) of [45, thm. 4.8], where one has to use [45, def. 4.2 and def. 4.3]. Definition [45, def. 4.2] is equivalent to our definition of "traceless" by Remark 2.27(viii). □

Part (ii) of Proposition 4.1, [44, thm. 5.9] and [45, prop. 5.14] together imply the following Corollary 4.4 immediately. But note that strongly purely infinite simple C^*-algebras are in general not approximately divisible, cf. [23].

Corollary 4.4. *An approximately divisible C^*-algebra A is locally purely infinite if and only if A is strongly purely infinite.*

Definition 4.5. We call a closed ideal J of a C^*-algebra D *stably generated* if there is a stable C^*-subalgebra E of D, which generates J as a closed ideal of D. (E can be assumed to be hereditary.)

Let I denote the ideal of D which is the is the closure of the upward directed net of the finite sums of stably generated ideals of D. We say that the set of stably generated closed ideals of D is *approximately upward directed*, if I is the closure of the union of stably generated ideals (union of sets, do not mix it up with sums of ideals). Equivalently this can be expressed as follows:

$$\text{(4.1)} \qquad \begin{cases} \text{If } D_1 \text{ and } D_2 \text{ are stable hereditary } C^*\text{-subalgebas of } D, \\ d_j(d_j)^* \in D_j \;\; (j = 1, 2) \text{ and } \delta > 0\,, \text{ then there exists a stable} \\ \text{hereditary } C^*\text{-subalgebra } D_3 \text{ of } D \text{ and } d_3 \in D\,, \text{ such that} \\ d_3(d_3)^* \in D_3 \text{ and } (d_3)^*d_3 = ((d_1)^*d_1 + (d_2)^*d_2 - \delta)_+. \end{cases}$$

Question 4.6. *Are the stably generated closed ideals of a traceless algebra D approximatively upward directed in the sense of Definition 4.5?*

Corollary 4.7. *Suppose that for every hereditary C^*-subalgebra D of A the stably generated closed ideals of D are approximately upward directed (in the sense of Definition 4.5).*

Then A is locally purely infinite if and only if A is purely infinite (=pi(1)).

Proof. Suppose A is l.p.i. Let $b \in A_+$ be a non-zero element and $\varepsilon > 0$. By Proposition 4.1 there are stable hereditary C^*-subalgebras $D_1, \ldots, D_n \subset D := \overline{bAb}$ such that $(b - \varepsilon/8)_+$ belongs to the closed ideal of D generated by $D_1 \cup \ldots \cup D_n$. Thus, there are $d_1, \ldots, d_n \in D$ such that $d_j(d_j)^* \in D_j$ and $d_1^*d_1 + \cdots + d_n^*d_n = (b - \varepsilon/4)_+$. One can find inductively from property (4.1) of Definition 4.5 some stable hereditary C^*-subalgebra $E \subset D$ and $d \in D$ with $dd^* \in E$ with $d^*d = (b - \varepsilon/2)_+$. Take isometries $s_1, s_2 \in \mathcal{M}(E)$ generating a copy of $\mathcal{O}_2$ and let $e = s_1 d$ and $f = s_2 d$. The row $g := (e, f) \in M_{1,2}(D)$ satisfies $g^*g = (b - \varepsilon/2)_+ \otimes 1_2$. Thus, b is properly infinite and A is purely infinite by Remark 2.9.

Conversely, if A is purely infinite then Lemma 2.10 applies to every non-zero element $a \in A_+$. □

A trivial consequence of Corollary 4.7 is the following corollary.

Corollary 4.8. *If the lattice of closed ideals of A is linearly ordered then A is locally purely infinite if and only if A is purely infinite.*

Now we are going to show that weak pure infiniteness implies local pure infiniteness. We need Lemma 2.10 and the following lemma for the proof.

Lemma 4.9. *Suppose that A satisfies part (i) of Definition 1.2 of pi(m). Then*

(i) *every quotient A/J and every hereditary C^*-subalgebra D of A satisfy part (i) of Definition 1.2,*

(ii) *$\ell_\infty(A)$ and the ultrapowers A_ω satisfy part (i) of Definition 1.2,*

(iii) *if non-zero $f_2, \ldots, f_{m+1} \in A$ satisfy the relations (2.4) for $n := m+1$, then*

$$a := f_2 f_2^* + \ldots + f_{m+1} f_{m+1}^*$$

satisfies the assumption of Lemma 2.10.

(iv) *every irreducible representation $d\colon A \to \mathcal{L}(\mathcal{H})$ of dimension $> m$ does not contain any non-zero compact operators in its image.*

Proof. (i): If $b \in (A/J)_+$ is in the closed ideal generated by $\pi_J(a) = a + J$ for $a \in A_+$, and if $\delta > 0$, then there are $f_1, \ldots, f_n \in A$ such that $(b - \delta)_+ = \pi_J(\sum_{1\le k\le n} f_k^* a f_k)$. On the other hand, there are $g_1, \ldots, g_m \in A$ with $\sum_{1\le j\le m} g_j^* a g_j = (c-\delta)_+$ for $c := \sum_{1\le k\le n} f_k^* a f_k$. Thus, $\|b - \sum d_j^* \pi_J(a) d_j\| < 2\delta$ for $d_j = g_j + J$, $j = 1, \ldots, m$.

The inequality $\|b - \sum d_i^* a d_i\| < \varepsilon$ implies $\|b - \sum (a^t d_i b^t)^* a (a^t d_i b^t)\| < \varepsilon$. for suitable $t > 0$. This shows that property (i) of Definition 1.2 of pi(m) passes to hereditary C^*-subalgebras.

(ii): Let $\varepsilon > 0$, $a, b \in \ell_\infty(A)_+$, such that b is in the closed ideal generated by a. There are $\eta > 0$ and $e^{(1)}, \ldots, e^{(p)} \in \ell_\infty(A)$ with $\sum (e^{(j)})^* (a-\eta)_+ e^{(j)} = (b - \varepsilon/2)_+$, cf. subsection 2.7. Thus, the n-th components $(b_n - \varepsilon/2)_+$ of $(b-\varepsilon/2)_+$ are in the ideal generated by the n-th components $(a_n - \eta)_+$ of $(a-\eta)_+$. If we apply part (i) of Definition 1.2 and Remark 2.7, then we get a column $f_n \in M_{m,1}(A)$ with $f_n^*((a_n - \eta)_+ \otimes 1_m) f_n = (b_n - \varepsilon)_+$. Let $d_n := (h(a_n) \otimes 1_m) f_n \in M_{m,1}$, where $h(t) = 0$ for $t \le \eta$ and $h(t) = ((t-\eta)/t)^{1/2}$ for $t > \eta$. Then $d_n^*(a_n \otimes 1_m) d_n = (b_n - \varepsilon)_+$, $\|d_n\|^2 \le \eta^{-1} \|b_n\|$, and $d \in M_{m,1}(\ell_\infty(A)) \cong \ell_\infty(M_{m,1}(A))$ with components d_n satisfies $d^*(a \otimes 1_m) d = (b - \varepsilon)_+$.

Since A_ω is a quotient of $\ell_\infty(A)$, it also satisfies condition (i) of Definition 1.2.

(iii): Let $f_1 := (f_2^* f_2)^{1/2}$ and take the polar decompositions $f_j = v_j f_1$ in A^{**} of f_j for $j = 2, \ldots, m+1$. Then $(a - \nu)_+ = g_2 g_2^* + \ldots + g_{m+1} g_{m+1}^*$ for $g_j := v_j (f_1^2 - \nu)^{1/2}$, $g_2, \ldots, g_{m+1}$ are in A and satisfy the relations (2.4). Thus, it suffices to show that a is properly infinite.

Let J be a closed ideal of A which does not contain a. Let $b := a + J$ and $h_j := f_j + J$. Then h_1 is non-zero in A/J, and $b + h_1$ is contained in the closed ideal generated by h_1^2. Since A/J satisfies again the property (i) of Definition 1.2, we find for every $\delta > 0$ elements $d_1, \ldots, d_m \in A/J$ with $\|b + h_1 - \sum d_j^* h_1^2 d_j\| < \delta$. Since $h_1 b = 0$, there is a row-contraction $c = (c_1, c_2) \in M_{1,2}(A)$ with $\|c^*(b + h_1)c - (b \oplus h_1)\| < \delta$. Hence $e_i := \sum_{1\le j\le m} f_{j+1} d_j c_i$ defines a row $e = (e_1, e_2) \in M_{1,2}(A)$ with $\|e^* b e - (b \oplus h_1)\| < 2\delta$. This shows that a is properly infinite by [44, prop. 3.14].

(iv): Since $d\colon A \to \mathcal{L}(\mathcal{H})$ is irreducible, $\mathcal{K}(\mathcal{H}) \subset d(A)$ or $\mathcal{K}(\mathcal{H}) \cap d(A) = 0$, i.e., the image of an irreducible representation d of A has non-zero intersection with the compact operators if and only if $\mathcal{K}(\mathcal{H})$ is a quotient of an ideal of A. Thus, $\mathcal{K}(\mathcal{H})$ must satisfy pi(m)(i) by part(i). But this means that $\mathcal{K}(\mathcal{H})$ can not contain a projection of rank $m+1$, because such a projection must be properly infinite in $\mathcal{K}(\mathcal{H})$ by (iii). Therefore $\mathcal{H}$ has dimension $\leq m$. □

Proposition 4.10. *If A is pi(m), then $\ell_\infty(A)$, every quotient of A, every hereditary C^*-subalgebra of A and the ultrapowers A_ω are pi(m).*

Proof. Parts (i) and (ii) of Lemma 4.9 say that property (i) of the Definition 1.2 of pi(m) passes to $\ell_\infty(A)$, A_ω and to hereditary C^*-subalgebras D of quotients A/J.

Since $\ell_\infty(\ell_\infty(A)) \cong \ell_\infty(A)$, we have that $\ell_\infty(A)$ also satisfies (ii) of Definition 1.2 and is pi(m).

Part (iv) of Lemma 4.9 implies that the images of irreducible representations of $\ell_\infty(A)$ have zero intersection with the compact operators, i.e., every quotient of $\ell_\infty(A)$ is antiliminal. $\ell_\infty(A_\omega)$ is a quotient of $\ell_\infty(\ell_\infty(A))$. It follows that the quotients $\ell_\infty(A/J)$ and $\ell_\infty(A_\omega)$ and the hereditary C^*-subalgebra $\ell_\infty(D)$ of $\ell_\infty(A)$ can not have a quotient of finite dimension. Thus, D, A/J and A_ω also satisfy condition (ii) of Definition 1.2. □

Proposition 4.11. *Let A be a C^*-algebra which satisfies condition (i) of the Definition 1.2 of pi(m) and which has no irreducible representation of dimension $\leq m$. Then A is locally purely infinite.*

In particular, every weakly purely infinite C^*-algebra is locally purely infinite.

Proof. Let J be a primitive ideal in A and let $b \in A_+ \setminus \{0\}$ be a positive element with $\|b+J\| > 0$. Let us construct a non zero stable C^*-algebra D in the hereditary C^*-subalgebra $B := \overline{bAb}$ such that $D \not\subset J$.

Let $d\colon A \to \mathcal{L}(\mathcal{H})$ be an irreducible representation with kernel equal to J. By (iii) of Lemma 4.9, we have $d(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ because the dimension of $\mathcal{H}$ is $> m$. In particular, $b\mathcal{H}$ has infinite dimension. Since B is hereditary in A $d_1\colon\, c \in B \mapsto d(c)|\overline{b\mathcal{H}}$ is also an irreducible representation of B of infinite dimension. There is a non-zero *-homomorphism h from $C_0(0,1] \otimes M_{m+1}(\mathbb{C})$ to B with $d_1 h \neq 0$, thanks to a variant of Glimm's classical lemma, see end of subsection 2.3. Therefore we find contractions $f_2, \ldots, f_{m+1}$ in $B \setminus J$ which satisfy the relations (2.4) for $n = m+1$. By part (iii) of Lemma 4.9, the sum $a := f_2 f_2^* + \ldots + f_{m+1} f_{m+1}^*$ satisfies that $(a-\nu)_+$ is properly infinite for all $\nu \in (0, \|a\|)$. Take $\varepsilon \in (0, \|a+J\|)$, then $(a-\varepsilon)_+ \notin J$ and Lemma 2.10 defines a stable hereditary C^*-subalgebra $\overline{dAd}$ of $B \subset A$ whose image in A/J is non-zero. □

The notion of pi(n) was introduced in [38] for a study of ultra-powers of approximately inner completely positive contractions on C^*-algebras, cf. [38]. The next proposition shows that our definition of pi(n) is formally weaker than [45, def. 4.3] but it shows also that the corresponding definitions of weakly purely infinite algebras are equivalent.

Proposition 4.12. *Let A be a weakly purely infinite C^*-algebra. Then there exists an integer $n > 0$ such that for every $a \in A_+ \setminus \{0\}$, the element $a \otimes 1_n \in M_n(A)$ is properly infinite.*

By [45, lemma 4.7] one has $m \le n$ if m is the smallest $m \in \mathbb{N}$ such that A is pi(m) in the sense of our Definition 1.2.

Proof. Suppose that A is pi(m) and take a free ultrafilter ω. By Proposition 4.10, the C^*-algebra A_ω also is pi(m). Proposition 4.11 implies that A_ω is l.p.i. Therefore there is no non-trivial lower semi-continuous 2-quasi-trace on A_ω by part (ii) of Proposition 4.1. But this means by Proposition 4.3 that A is weakly purely infinite in the sense of [45, def. 4.3]. $\square$

Remarks 4.13. Summing up, we have also shown that for the ultrapowers A_ω of A holds: $\quad A_\omega$ *is traceless* $\iff$ A_ω *is l.p.i.* $\iff$ A_ω *is w.p.i.*

If A is w.p.i. and C is a finitely generated commutative C^*-subalgebra of A_ω then the he relative commutant $C' \cap A_\omega$ is w.p.i. : it is application of Lemma 2.5 and of ideas from [45], see [40].

If one could show, that $a \otimes 1_{m^2}$ is properly infinite for every non-zero element a in a stable C^*-algebra B, provided this happens for every non-zero a in a closed ideal J of B and for every non-zero element a of B/J, then one would get (by part (iii) of Lemma 4.9, [45] and the local Glimm halving lemma) that, conversely, every element a in a C^*-algebra A with property (i) of Definition 1.2 and without irreducible representations of dimension $\le m$ satisfies that $a \otimes 1_{m^2}$ is purely infinite.

Thus, $\ell_\infty(A)$ could be replaced by A itself in part (ii) of Definition 1.2.

It is still unknown (in 2003) whether any pi(n) C^*-algebra (in the sense of Definition 1.2) is automatically pi(1), i.e., purely infinite. However the Glimm halving property 2.6 yields the following.

Proposition 4.14. *Suppose that a C^*-algebra A satisfies property (i) of Definition 1.2 of pi(m). Then A is pi(1) if and only if A has the global Glimm halving property 2.6.*

Proof. As we have noticed in part (iv) of Remark 2.9, every p.i. algebra has the global Glimm halving property. C^*-algebras A with the global Glimm halving property have only anti-liminal (non-zero) quotients, in particular A has no character.

Given two positive elements $a, b \in A_+$ such that b is in the closed ideal generated by a and $\varepsilon > 0$, let us construct $d \in A$ with $\|d^*ad - b\| < 2\varepsilon$: by Remark 2.9(ii), there exist $\delta > 0$ and $c_1, \ldots, c_m \in A$, such that

$$\sum_{1 \le k \le m} c_k^*(a - 3\delta)_+ c_k = (b - \varepsilon)_+.$$

The global Glimm halving property yields the existence of $f_1, \ldots, f_m$ in the closure of $(a - 2\delta)_+ A (a - 2\delta)_+$, such that $f_i^* f_j = \delta_{ij} f_1^* f_1$ and $(a - 3\delta)_+$ belongs to the ideal of A generated by $f_0 := f_1^* f_1$ (cf. the remark following Definition 2.6). Thus, $(b - \varepsilon)_+$ is in the ideal generated by f_0. As A is pi(m), there exist $d_1, \ldots, d_m \in A$ such that

$$\| \sum_{1 \le j \le m} d_j^* f_0 d_j - (b - \varepsilon)_+ \| < \varepsilon.$$

Let $g_\delta \in C_0((0, \infty])$ be as in (2.7) and let $h_\delta(t) := (g_\delta(t)/t)^{1/2}$. Then $g_\delta(a) f_i = f_i$, and $d := h_\delta(a) \sum_{1 \le j \le m} f_j d_j$ satisfies $\|d^*ad - b\| = \| \sum_{i,j} d_i^* f_i^* g_\delta(a) f_j d_j - b\| < 2\varepsilon$. $\square$

A projections q in a C^*-algebra A is *properly infinite* (respectively *infinite*) if there are partial isometries $u, v \in A$ with $u^*u = v^*v = q$ and $vv^* + uu^* \leq q$ (respectively there is a partial isometry $v \in A$ with $v^*v = q$, $vv^* \leq q$ and $vv^* \neq q$). q is *full* if A is the closed two-sided ideal of A generated by q. A result of Cuntz in [17] can be expressed equivalently as follows:

Lemma 4.15. *If a C^*-algebra C contains a properly infinite full projection, then every element $z \in K_0(C)$ is represented by a full and properly infinite projection q in C, i.e., $z = [q]$, and two properly infinite full projections $q, q' \in C$ define the same element $[q] = [q']$ of $K_0(C)$ if an only if q and q' are Murray–von Neumann equivalent, i.e., there is a partial isometry $v \in C$ with $v^*v = q$ and $vv^* = q'$, denoted: $q \sim_v q'$.*

If q, r are properly infinite full projections in C, then one can find properly infinite full projections $q' \sim q$ and $r' \sim r$ in C which are orthogonal, i.e., $r'q' = 0$. The sum $y := [q] + [r] \in K_0(C)$ is represented by $y = [q' + r']$.

*In particular, the neutral element $0 \in K_0(C)$ is represented by a full projection $p \in C$, such that there is a *-monomorphism $\psi\colon \mathcal{O}_2 \to C$ with $\psi(1) = p$.*

Lemma 4.16. *Suppose that B is stably isomorphic to a unital C^*-algebra A. Then the following are equivalent:*

(i) *Every semi-finite lower semi-continuous 2-quasi-trace on B is zero.*
(ii) *There is a *-monomorphism $\psi\colon \mathcal{O}_2 \to B \otimes \mathcal{K}$ such that $\psi(1)$ generates $B \otimes \mathcal{K}$ as a two-sided ideal.*

The assumption that B is stably isomorphic to a unital C^*-algebra is implied by (i) alone if B is simple and σ-unital, cf. [5].

Proof. (i)⇒(ii): Let $C := B \otimes \mathcal{K}$. Then B is isomorphic to a "full" corner D of $A \otimes \mathcal{K}$, because $C \cong A \otimes \mathcal{K}$. Every finite lower semi-continuous 2-quasi-trace τ on A_+ extends uniquely to a semi-finite lower semi-continuous 2-quasi-trace ρ on $(A \otimes \mathcal{K})_+$ with $\tau(a) = \rho(a \otimes e_{1,1})$ for $a \in A_+$. The restriction $\rho|D$ of ρ to D is again a semi-finite lower semi-continuous 2-quasi-trace with $D \cap I_\rho = I_{(\rho|D)}$, cf. Definition 2.22 and Remarks 2.27(viii). Thus, our assumption (i) implies $\rho|D = 0$, $D \subset I_\rho$. Since D is full and I_ρ is a closed ideal, it follows $\rho = 0$ and $\tau = 0$. Thus, every finite 2-quasi-trace on A is equal to zero. Let 1_A be the unity element of A. By [44, prop. 5.7] we find $k \in \mathbb{N}$ such that $(1_A \otimes 1_k) \oplus (1_A \otimes 1_k)$ is equivalent to a sub-projection of $1_A \otimes 1_k$ in $A \otimes M_{2k}$. Thus, $1_A \otimes 1_k$ defines a properly infinite projection r of C which is a full projection in C, i.e., the ideal generated by r is dense in C. By Lemma 4.15 there is a *-homomorphism $\psi\colon \mathcal{O}_2 \to C = B \otimes \mathcal{K}$ such that $\psi(1)$ is full in C.

(ii)⇒(i): For every semi-finite lower semi-continuous 2-quasi-trace τ on B_+ there is a unique semi-finite lower semi-continuous 2-quasi-trace ρ on $(B \otimes \mathcal{K})_+$ with $\tau(b) = \rho(b \otimes e_{1,1})$. Since $\psi(1)$ is in the positive part of the Pedersen ideal of $B \otimes \mathcal{K}$, we must have $\rho(\psi(1)) < \infty$. Thus, the quasi-trace $\rho\psi$ on $\mathcal{O}_2$ must be zero, and therefore ρ is zero on the closed ideal generated by $\psi(1)$ (which is contained in I_ρ). □

Theorem 4.17. *Let A be C^*-algebra of real rank zero.*

Then A is locally purely infinite if and only if A is strongly purely infinite.

Proof. Since s.p.i. implies p.i., it implies also l.p.i. by Proposition 4.11. Conversely, A is s.p.i. if A is p.i. by [45, cor. 6.9], and it suffices to show that A is p.i. if A is l.p.i.

By [44, prop. 4.7], A is purely infinite if every non-zero hereditary C^*-subalgebra D in every quotient A/I of A contains an infinite projection.

A/I and D are again l.p.i. by part (iii) of Proposition 4.1. D contains a (non-zero) stable C^*-subalgebra E by Proposition 4.1 (i). Upon replacing E by EDE, we may assume that E also is hereditary in A/I and, thus, E is l.p.i. by Proposition 4.1 (iii).

A/I and E have real rank zero if A has real rank zero, and E contains an approximate unity consisting of projections, see [13].

If $0 \neq q \in E$ is a projection, then qEq is a unital l.p.i. C^*-algebra by Proposition 4.1 (iii) and qEq is traceless by Proposition4.1 (ii).

By Lemma 4.16, $qEq \otimes \mathcal{K} \subset E \otimes \mathcal{K} \cong E \subset D$ contains a properly infinite projection. □

5. Infiniteness and Hausdorff primitive ideal spaces

Proposition 5.1. *Let A be a C^*-algebras with Hausdorff primitive ideal space.*

Then A is l.p.i. if and only if every simple quotient of A is p.i.

Proof. Let J be a primitive ideal of A. Then the quotient A/J is simple, because $Prim(A)$ is Hausdorff. If A is l.p.i. then every simple quotient is l.p.i., hence p.i. by Proposition 3.1.

Suppose conversely that A/J is p.i. for every primitive ideal J of A. Let $b \in A_+ \setminus \{0\}$ and J a primitive ideal of A which does not contain b. We construct a stable hereditary C^*-algebra contained in $D := \overline{bAb}$ but which is not contained in J: by (i) and (ii) of Proposition 3.1, there exists a properly infinite projection, whence (by Lemma 4.15) a copy of $\mathcal{O}_2$, in the quotient $B := D/(J \cap D) = \pi_J(D)$. But $\mathcal{O}_2$ is semi-projective and so, by Remark 2.4 there exist a closed neighborhood F of J in $X := \mathrm{Prim}(D) \subset \mathrm{Prim}(A)$ and a *-homomorphism $\psi\colon \mathcal{O}_2 \to D_{|F}$ such that $\psi(1)$ generates $D_{|F}$ as two-sided ideal. Since the interior of Z contains J, we find a function $h \in C_0(X)$ with $0 \leq h(I) \leq h(J) = 1$ for all primitive ideals $I \in X$, such that the support of h is contained in F. Then $f \otimes c \mapsto f(h)\psi(c) \in D$ extends to a monomorphism $\varphi\colon C_0(Y) \otimes \mathcal{O}_2 \to D$ where $Y = h(X) \setminus \{0\}$. The image of this monomorphism is not contained in J, because $\pi_J \varphi$ has kernel $C_0(Y \setminus \{1\}) \otimes \mathcal{O}_2$. The image of φ contains a stable C^*-subalgebra which is also not contained in J, because $\mathcal{O}_2$ contains a copy of $\mathcal{K}$. □

Proposition 5.2. *Let A be a C^*-bundle over a Hausdorff locally compact space X with finite topological dimension n and suppose there is an integer $k > 0$ such that every fiber A_x ($x \in X$) satisfies condition (i) of Definition 1.2 for $m = k$.*

Then A satisfies condition (i) of Definition 1.2 for $m = k(1 + n)$.

If, moreover, the C^*-algebra A has the global Glimm halving property 2.6, then A is purely infinite (= pi(1)).

Proof. Let $a, b \in A_+$ be positive elements with b in the closed ideal generated by a, and $\varepsilon > 0$. The function $h(x) := g_\delta(\|b_x\|)^{1/2}$ (with g_δ as in (2.7) and $\delta = \varepsilon/2$) is a continuous function on X with compact support and satisfies $\|h^2b - b\| = \sup_X(1-h^2)N(b) < \varepsilon$, because the fiber norm function $x \mapsto N(b)(x) := \|b_x\|$ is in $C_0(X)_+$. One can find for each point x in the compact closure F of $\{x \in X\,;\, h(x) > 0\}$ a column-matrix $d(x) \in M_{k,1}(A)$ such that $\|[b - d(x)^*(a\otimes 1_k)d(x)]_x\| < \varepsilon/2$, whence by upper semi-continuity of the norm-functions there is an open neighborhood $U_x \ni x$ on which $\|[b - d(x)^*(a\otimes 1_k)d(x)]_y\| < \varepsilon$ for all y in U_x.

There is a finite open covering $\mathcal{U} = \{U_1, \ldots, U_p\}$ of F and elements $d_j \in M_{k,1}(A)$ satisfying $\|[b - (d_j)^*(a\otimes 1_k)d_j]_y\| < \varepsilon$ for all y in U_j, where $1 \le j \le p$. By Lemma 2.5, one can moreover assume, up to taking a suitable refinement of $\mathcal{U}$, that there exits a map $\iota\colon \{1,\ldots,p\} \to \{1,\ldots,n+1\}$ such that for each $1 \le i \le n+1$, the open set

$$Y_i = \bigcup_{j\in\iota^{-1}(i)} U_j$$

is the *disjoint* union of the open sets U_j, $j \in \iota^{-1}(i)$, because F has dimension $\le n$. Now take $e_j \in C_0(U_j)_+ \subset C_0(X)$ with $\sum_{1\le j\le p} e_j \le 1$ and $(\sum e_j)_{|F} = 1$, and define, for $i \in \{1,\ldots,n+1\}$, $\eta_i := \sum_{j\in\iota^{-1}(i)} e_j$ and

$$d^{(i)} := \sum_{j\in\iota^{-1}(i)} (e_j)^{1/2}\, d_j \in M_{k,1}(A).$$

Then $\|\,[\eta_i b - (d^{(i)})^*(a\otimes 1_k)d^{(i)}]_y\| < \eta_i(y)\varepsilon$ if $\eta_i(y) > 0$ and $1 \le i \le n+1$.

Thus, $\|\,b - f^*(a\otimes 1_{k(n+1)})f\| \le 2\varepsilon$ for the column $f \in M_{k(n+1),1}(A)$ with $f_{ik+j,1} := h.(d^{(i)})_j$, because $h^2\sum \eta_i = h^2$.

If A satisfies, in addition, the global Glimm halving property then Proposition 4.14 applies. □

Corollary 5.3. *A C^*-algebra A with Hausdorff finite dimensional primitive ideal space X is purely infinite if and only if it is locally purely infinite.*

Proof. $\Rightarrow$: The property pi(1) always implies l.p.i. by Proposition 4.11.

$\Leftarrow$: Conversely, by Proposition 5.1, the simple quotients A/J are purely infinite if A is locally purely infinite. In particular A/J is anti-liminal.

Since, by assumption, $X := \mathrm{Prim}(A)$ is Hausdorff, every primitive quotient of A is simple and A is a C^*-bundle over X.

Since X has finite dimension and the fibers A/J are pi(1), we get from Proposition 5.2 and from Theorem 2.7 that A satisfies condition (i) of Definition 1.2 of pi(m) for $m = 1 + \mathrm{Dim}(X)$ and that A has the global Glimm halving property.

Thus, Proposition 4.14 implies that A is purely infinite. □

It is unknown (2003) whether any purely infinite C^*-algebra A is strongly purely infinite, but we provide below a positive answer in the case when the primitive ideal space $\mathrm{Prim}(A)$ is Hausdorff (Theorem 5.8).

Before that we show a result of independent interest, Proposition 5.6. It is an appropriate generalization for σ-unital C^*-algebras A with Hausdorff $\mathrm{Prim}(A)$ of a theorem of Blackadar and Cuntz [5] which says that stable simple C^*-algebras without any non-trivial lower semi-continuous quasi-trace contain a properly infinite projection.

Lemma 5.4. *Let h and k be unital *-homomorphisms from the Cuntz algebra $\mathcal{O}_2$ into a unital C^*-algebra B. Then there is a norm-continuous map $U\colon t \in [0,+\infty) \mapsto U(t)$ into the unitary group of B, such that, for $a \in \mathcal{O}_2$,*

$$k(a) = \lim_{t\to\infty} U(t)^* h(a) U(t).$$

In particular, there exists a unitary $U \in B$ and a selfadjoint element $b \in B$, such that $U^*h(s_i)U = e^{ib}k(s_i)$, $i = 1,2$ for the canonical generators s_1, s_2 of $\mathcal{O}_2$.

Note that in general U can not be found in the connected component of 1_B (Consider the Calkin algebra $B = \mathcal{L}(\ell_2)/\mathcal{K}(\ell_2)$ and examine the indices of the unitary operators of form $h(s_1)k(s_1)^* + h(s_2)k(s_2)^*$). A part of Lemma 5.4 has been proven by Rørdam [55, thm. 3.6] in the case where the logarithmic length of the connected component of the unitary group of B is finite (which is not the case for general C^*-algebras). We deduce the general result from the independent result [38, thm. B]:

Proof. Denote by $h_0 := h \otimes id_{\mathcal{K}}$, $h_1 := k \otimes id_{\mathcal{K}}$ and $h_2 := id_{\mathcal{O}_2} \otimes id_{\mathcal{K}}$ the stabilization of h, k and $id_{\mathcal{O}_2}$, respectively. Since $KK(\mathcal{O}_2, B) = 0$, it follows from [38, thm. B(ii)] that $h_i \oplus h_i$ is unitarily homotopic to $h_i \oplus h_j$, $i,j = 0,1$. By [38, thm. B(iii)], h_2 is unitarily homotopic to $h_2 \oplus h_2$. This implies that h_i is unitarily homotopic to $h_i \oplus h_i$, $i = 0,1$. It means $h_1(d) = \lim V(t)^* h_0(d) V(t)$ for a norm-continuous map V from $\mathbb{R}_+$ into the unitary group of the multiplier algebra of $B \otimes \mathcal{K}$. In particular, $1_B \otimes e_{1,1} = \lim V(t)^*(1_B \otimes e_{1,1})V(t)$. Thus, for large $t \in \mathbb{R}_+$ we can take a small correction of $V(t)$, such that our new $V(t)$ commutes with $1 \otimes e_{1,1}$. Thus, after a re-parameterization and a small perturbation, we may assume that $V(t) = U(t) \otimes e_{1,1} + W(t)$ with $W(t)^*W(t) = W(t)W(t)^* = 1 \otimes (1 - e_{1,1})$. Then $t \mapsto U(t) \in B$ has the desired property.

In the second statement we can take $U := U(t)$ for some large $t \in \mathbb{R}_+$ such that $Z := U^*h(s_1)Uk(s_1)^* + U^*h(s_2)Uk(s_2)^*$ is a unitary with distance < 1 from 1_B. Let $b := -i \log Z$. ☐

Proposition 5.5. (i) *Suppose that B is a stable C^*-algebra, J is a closed ideal of B, and that $\psi\colon \mathcal{O}_2 \to B$ and $\lambda\colon \mathcal{O}_2 \to B/J$ are *-homomorphisms.*

*If $\pi_J\psi(1)$ and $\lambda(1)$ generate the same closed ideal of B/J, then there is a *-homomorphism $\varphi\colon \mathcal{O}_2 \to B$, such that $\pi_J\varphi = \lambda$, and that $\varphi(1)$ and $\psi(1)$ generate the same closed ideal of B.*

(ii) *Suppose that the *-morphisms $\eta_k\colon A \to B_k$ ($k = 1,2$) define A as a pullback of the epimorphism $\pi_k\colon B_k \to C$, ($k = 1,2$), and that B_1, B_2 are stable.*

*If $\varphi_k\colon \mathcal{O}_2 \to B_k$ are *-morphisms such that $\pi_1\varphi_1(1)$ and $\pi_2\varphi_2(1)$ generate the same ideal of C, then there exists a *-morphism $\psi\colon \mathcal{O}_2 \to A$ such that $\eta_1\psi = \varphi_1$ and that $\eta_2\psi(1)$ and $\varphi_2(1)$ generate the same ideal of B_2.*

Proof. (i): $p := \pi_J\psi(1)$ and $q := \lambda(1)$ are properly infinite projections which generate the same closed ideal C of B/J, i.e., p and q are full properly infinite projections. Since $[p] = [q] = 0$ in $K_0(C)$, p and q are Murray–von Neumann equivalent by Lemma 4.15, i.e., there is a partial isometry $w \in C \subset B/J$ with $w^*w = p$ and $ww^* = q$.

We can find $a, b, c, \in B$ with $\pi_J(a) = w$, $\pi_J(b) = \lambda(s_1)$ and $\pi_J(c) = \lambda(s_2)$, where s_1 and s_2 denote the canonical generators of $\mathcal{O}_2$. Let $d := \psi(s_1)$, $e := \psi(s_2)$.

Since B is stable, the separable C^*-subalgebra of B which is generated by $\{a, b, c, d, e\}$ is contained in a separable and stable C^*-subalgebra B_1 of B. The C^*-algebra B_1 contains $\psi(\mathcal{O}_2)$, and the image of B_1 in B/J contains w and $\lambda(\mathcal{O}_2)$, and is naturally isomorphic to B_1/J_1, where $J_1 := B_1 \cap J$.

Thus, to prove (i), we can in addition assume, that B itself is separable (and stable by assumption).

Then B/J must be stable and separable, and the Murray–von Neumann equivalence of p and q implies the existence of a unitary $W \in \mathcal{M}(B/J)$ with $W^*pW = q$. An elementary matrix construction argument (which the the reader can find in K-theory textbooks) shows that the unitary W can be chosen in the connected component of the identity element of $\mathcal{U}(\mathcal{M}(B/J))$. (Here one could also use the generalized Kuiper theorems of Cuntz–Higson [19] or Mingo [50], which say that the unitary group of $\mathcal{M}(D)$ for a stable σ-unital C^*-algebra D is norm-contractible.)

The separability of B implies that the natural strictly continuous *-homomorphism $\mathcal{M}(\pi_J)$ from $\mathcal{M}(B)$ into $\mathcal{M}(B/J)$ is an epimorphism, [51, prop. 3.12.10]. Thus, we find a unitary V in $\mathcal{M}(B)$ with $\mathcal{M}(\pi_J)(V) = W$. Let $r := V^*\psi(1)V$, $\psi_1(d) := V^*\psi(d)V$. Then, $r = \psi_1(1)$ is Murray–von Neumann equivalent to $\psi(1)$, and $\pi_J(r) = q = \lambda(1)$. We find a partial isometry $d \in B$ such that $d^*d = r$ and $rd = 0$, because B is stable.

By Lemma 5.4 we can find in the unital C^*-algebra $q(B/J)q$ a unitary u and a selfadjoint b with $u^*\lambda(s_j)u = e^{ib}\pi_J\psi_1(s_j)$ for $j = 1, 2$. Let $z \in rBr$ be a contractive lift of u and let $c \in rBr$ be a selfadjoint lift of b. We obtain a new unital *-homomorphism $\psi_2\colon \mathcal{O}_2 \to rBr$ by $\psi_2(s_j) := e^{ic}\psi_1(s_j)$ for $j = 1, 2$.

$$U_0 := z + (r - zz^*)^{1/2}d^* - d(r - z^*z)^{1/2} + dz^*d^*$$

is a unitary in $(r + dd^*)B(r + dd^*)$. We define $\varphi(d) := U_0\psi_2(d)U_0^*$. Then $\pi_J\varphi = \lambda$, and $\varphi(1)$ is Murray–von Neumann equivalent to r.

(ii): The pullback condition says that $\eta\colon a \mapsto (\eta_1(a), \eta_2(a))$ is an isomorphism from A onto the C^*-subalgebra $\{(b_1, b_2);\ b_j \in B_j, \pi_1(b_1) = \pi_2(b_2)\}$ of $B_1 \oplus B_2$. By (i) there exist $\varphi_3\colon \mathcal{O}_2 \to B_2$ such that $\pi_2\varphi_3 = \pi_1\varphi_1$ and that $\varphi_3(1)$ and $\varphi_2(1)$ generate the same ideal of B_2. $\psi(d) := \eta^{-1}(\varphi_1(d), \varphi_3(d))$ for $d \in \mathcal{O}_2$ is as desired. □

Proposition 5.6. *Assume that A is a σ-unital C^*-bundle over a Hausdorff space X and that for every $y \in X$, there is a properly infinite and full projection q_y in $A_y \otimes \mathcal{K}$.*

Then there exists a non-degenerate $C_0(X)$-linear monomorphism

$$h_0\colon C_0(X) \otimes \mathcal{O}_2 \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}\,.$$

Recall that a projection p in a C^*-algebra B is *full* if the span of BpB is dense in B.
Proof. The positive part A_+ contains a strictly positive element e, because A is σ-unital. We can assume $\|e\| > 1$. The function $N(e) \in C_0(X)_+$ must satisfy $N(e)(y) > 0$ for every $y \in X$. Let $Y_n := \{y \in X;\ N(e)(y) \geq 1/n\}$. Then Y_n is compact, is contained in the interior of Y_{n+1}, and $\bigcup Y_n = X$.

Since $\mathcal{K}$ is exact and simple, $A \otimes \mathcal{K}$ is again a C^*-bundle over X with fibers $A_x \otimes \mathcal{K}$, by [46], and $A \otimes \mathcal{K}$ is again σ-unital.

Thus, we can replace A by $A\otimes\mathcal{K}$ and can assume from now that A is moreover stable and therefore has stable fibers A_y for $y \in X$. It follows that A_y contains a properly infinite full projection q_y. (We have even $A_y q_y A_y = A_y$ because q_y is properly infinite.)

Note that properly infinite projections must be non-zero. The existence of $q_y \neq 0$ in $A_y \otimes \mathcal{K}$ implies that $A_y \neq 0$ for every $y \in X$.

By Lemma 4.15 the class of the zero-element of $K_0(A_y)$ can be represented by $p_y = \psi_y(1)$, where $\psi_y \colon \mathcal{O}_2 \to A_y$ a *-monomorphism and p_y is a full projection of A_y.

Since $\mathcal{O}_2$ is semi-projective, by Remark 2.4 there is for every point $y \in X$ a compact neighborhood $F \subset X$ of y (i.e., in particular y is contained in the interior of F) and a *-homomorphism $\psi \colon \mathcal{O}_2 \to A_{|F}$ such that $\psi(1)$ generates $A_{|F}$ as a closed ideal, cf. subsection 2.4.

Now let F and G two compact subsets of X and $\psi_1 \colon \mathcal{O}_2 \to A_{|F}$, $\psi_2 \colon \mathcal{O}_2 \to A_{|G}$ *-homomorphisms, such that $\psi_1(1)$ generates $A_{|F}$ and $\psi_2(1)$ generates $A_{|G}$ as closed ideals. By Remark 2.3, $A_{|F\cup G}$ is the pull-back of $A_{|F}$ and $A_{|G}$ along $A_{|F\cap G}$. Thus, by (ii) of Proposition 5.5, there is a *-homomorphism $\psi \colon \mathcal{O}_2 \to A_{|F\cup G}$ such that $\psi(1)$ generates $A_{|F\cup G}$ as a closed ideal.

This shows that for every compact subset $Y \subset X$ there is a *-homomorphism $\psi \colon \mathcal{O}_2 \to A_{|Y}$ such that $\psi(1)$ generates $A_{|Y}$ as a closed ideal. Thus, we find monomorphisms $\psi_n \colon \mathcal{O}_2 \to A_n := A_{|Y_n}$, such that $\psi_n(1)$ generates A_n as a closed ideal.

Let now $\varphi_1 := \psi_1$ and assume that we have found $\varphi_j \colon \mathcal{O}_2 \to A_j$, $j = 1, \dots, n$ such that $\varphi_j(1)$ generates A_j as a closed ideal and that $\varphi_j(d) = \varphi_{j+1}(d)_{|Y_j}$ for $d \in \mathcal{O}_2$ and $j = 1, \dots, n-1$. Since A_{n+1} is stable, we can apply (i) of Proposition 5.5 to $\varphi_n \colon \mathcal{O}_2 \to A_n$, $\psi_{n+1} \colon \mathcal{O}_2 \to A_{n+1}$ and the natural epimorphism from A_{n+1} onto A_n, and get $\varphi_{n+1} \colon \mathcal{O}_2 \to A_{n+1}$ such that $\varphi_n(d) = \varphi_{n+1}(d)_{|Y_n}$ for $d \in \mathcal{O}_2$ and $\varphi_{n+1}(1)$ is full in A_{n+1}.

For every $f \in C_c(X)$ there is $n \in \mathbb{N}$ such that the closure of the support of f is contained in the interior of Y_n. Thus, we get a well-defined algebra *-homomorphism γ_0 from the algebraic tensor product $C_c(X) \odot \mathcal{O}_2$ into A which is given on elementary tensors $f \otimes d$ by $\gamma_0(f \otimes d) := f\varphi_n(d)$ for n sufficiently large. By construction it is a $C_c(X)$-linear map. It is well-known (and can easily be seen from [61] or [60, prop. 1.22.3]), that the universal C^*-hull of $C_c(X) \odot \mathcal{O}_2$ is naturally isomorphic to $C_0(X, \mathcal{O}_2)$. Thus, γ_0 extends to a *-homomorphism γ_1 from $C_0(X, \mathcal{O}_2)$ into A. The map γ_1 is $C_0(X)$-linear, and for every fiber A_x, $x \in Y_n$, the fiber *-homomorphism $(\gamma_1)_x = (\varphi_n)_x \colon \mathcal{O}_2 \to A_x$ is a monomorphism, because $\varphi_n(1)$ generates $A_{|Y_n}$ as a closed ideal. It follows that γ_1 is a $C_0(X)$-linear *-monomorphism from $C_0(X, \mathcal{O}_2)$ into A such that the image B of γ_1 generates A as a closed ideal.

Let D_1 be the hereditary C^*-subalgebra of $A \otimes \mathcal{K}$ which is generated by $B \otimes \mathcal{K}$. Then D_1 is stable, σ-unital and generates $A \otimes \mathcal{K}$ as an ideal. The same happens with $D_2 := A \otimes p_{11}$. A closer look to the proof of the stable isomorphism theorem of Brown [12] shows that it gives an element $a \in A \otimes \mathcal{K}$, such that a^*a is a strictly positive element of D_1 and aa^* is a strictly positive element of D_2. The polar decomposition $a = v(a^*a)^{1/2}$ in the second conjugate of $A \otimes \mathcal{K}$ induces a $C_0(X)$-linear isomorphism λ

from D_1 onto D_2 by $\lambda\colon\ a \mapsto vav^*$, see Remark 3.4. The $C_0(X)$-linear *-monomorphism $h_0 := \lambda \circ (\gamma \otimes \mathrm{id}_{\mathcal{K}})$ from $C_0(X, \mathcal{O}_2 \otimes \mathcal{K})$ into A is non-degenerate by construction. □

Corollary 5.7. *Suppose that A is a σ-unital C^*-algebra with Hausdorff primitive ideal space X and that A admits no non-zero semi-finite lower semi-continuous 2-quasi-trace.*

Then there exists a non-degenerate $C_0(X)$-linear monomorphism

$$h_0\colon C_0(X) \otimes \mathcal{O}_2 \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}\,.$$

Proof. The assumptions imply that for every point $x \in X$, the primitive quotient A_x must be simple and can not have a non-zero semi-finite lower semi-continuous quasi-trace. Thus, there is no non-zero semi-finite lower semi-continuous dimension function on $A_x \otimes \mathcal{K}$. Therefore $A_x \otimes \mathcal{K}$ contains a properly infinite projection q_x by [5]. Now Proposition 5.6 applies. □

Theorem 5.8. *Every purely infinite C^*-algebra A with Hausdorff primitive ideal space X is strongly purely infinite.*

Proof. By the permanence properties of p.i. and s.p.i. shown in [44] and [45] we can moreover assume that the p.i. C^*-algebra A is stable and σ-unital (and hence that X is σ-compact).

Then by [45, thm. 6.8], it is enough to prove that the C^*-algebra A has the *locally central decomposition property*, i.e., for every $a \in A_+$ and $\varepsilon > 0$, there exist $a_1, \ldots, a_n$ in A_+ such that

(i) each a_i is in the center of $\overline{a_i A a_i}$ $(1 \le i \le n)$,
(ii) $a_i \in \overline{\mathrm{span}(AaA)}$ $(1 \le i \le n)$,
(iii) $(a - \varepsilon)_+$ belongs to $\overline{A(\sum_i a_i)A}\,$.

By Corollary 5.7 there is a non-degenerate monomorphism $\pi\colon\ C_0(X) \otimes \mathcal{O}_2 \otimes \mathcal{K} \hookrightarrow A\,$. Then for each positive element $a \in A_+$, the operator $a_1 = \pi(N(a) \otimes 1_{\mathcal{O}_2} \otimes e_{1,1})$ has the expected properties since $N(a) = N(a_1)\,$. □

Remark 5.9. One can also directly prove that any locally purely infinite C^*-algebra A with Hausdorff primitive ideal space X has the locally central decomposition property thanks to Proposition 5.1: more generally, the locally central decomposition property holds for every C^*-bundle A over a locally compact space X with fibers A_x, such that every closed ideal J of A_x is generated by projections $p \in J$ (as a closed ideal of A_x). Because then the semi-projectivity of $\mathbb{C}$, exploited in the same way as the semi-projectivity of $\mathcal{O}_2$ in the proof of Proposition 5.6, gives that every ideal of A is generated by elements $b \in A_+$ of the form $b = fq$ where q is a projection in $A_{|F}$ for some compact subset $F \subset X$ with open interior and $f \in C_0(X)$ has support in F. Those elements b are trivially in the center of $\overline{bAb} \subset qA_{|F}q$.

Corollary 5.10. *Suppose that A and B are exact, have Hausdorff primitive ideal spaces of finite dimension, and that every simple quotient of A or of B is not isomorphic to the compact operators.*

Then $A \otimes B$ is s.p.i. if and only if $A \otimes B$ has no non-zero semi-finite lower semi-continuous trace.

Proof. We have seen the general implications s.p.i. $\Rightarrow$ p.i. $\Rightarrow$ traceless.

Since A and B are exact the primitive ideal space of $A \otimes B$ is natural isomorphic to the Tychonoff product of its primitive ideal spaces by part (2) of Proposition 2.17. Thus, $\mathrm{Prim}(A \otimes B)$ is Hausdorff of finite dimension. The simple quotients are the tensor products $(A/I) \otimes (B/J)$ of the simple quotients of A and B. As shown in the proof of part (ii) of Corollary 3.11, $(A/I) \otimes (B/J)$ is purely infinite if it has no non-zero semi-finite lower semi-continuous trace.

Thus, if $A \otimes B$ has no non-zero lower semi-continuous trace, then Proposition 5.1, Corollary 5.3 and Theorem 5.8 apply all to $A \otimes B$. $\square$

Let us finish with a local characterization of pure infiniteness for nuclear C^*-algebras.

Corollary 5.11. *Let A be a separable stable nuclear C^*-algebra whose primitive ideal space X is Hausdorff and of finite dimension.*

Then $A \cong A \otimes \mathcal{O}_\infty$ if and only if $A_x \cong A_x \otimes \mathcal{O}_\infty$ for every $x \in X$.

Proof. $\Leftarrow$: The primitive quotients A_x of A are simple if $\mathrm{Prim}(A)$ is Hausdorff. Thus, $A_x \cong A_x \otimes \mathcal{O}_\infty$ is purely infinite by Corollary 3.11, because $\mathcal{O}_\infty$ is purely infinite by [17]. Since $\mathrm{Prim}(A)$ is a Hausdorff space of finite dimension and A_x is purely infinite, we get from Corollary 5.3 that A is purely infinite, hence is strongly purely infinite by Theorem 5.8.

If A is moreover stable and nuclear, then [45, thm. 8.6] gives that A tensorially absorbs $\mathcal{O}_\infty$.

$\Rightarrow$: Conversely the isomorphism $A \cong A \otimes \mathcal{O}_\infty$, the exactness and the simplicity of $\mathcal{O}_\infty$ imply that every primitive quotient A_x of A must be isomorphic to $A_y \otimes \mathcal{O}_\infty$ for some $y \in \mathrm{Prim}(A)$.

Since $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ by [38, cor. H] (or by [47], [42], [45], or by a simple modification of [57]), this implies that $A_x \cong A_x \otimes \mathcal{O}_\infty$. $\square$

Remark 5.12. We do not known whether any continuous C^*-bundle A over the Hilbert cube $[0,1]^\infty$ with fibers isomorphic to $\mathcal{O}_2$ is necessarily purely infinite. (This question is open even if we suppose in addition that A is pi(2).) A negative answer would imply that l.p.i. algebras in general are not p.i. Then Question 4.6 also would have a negative answer.

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